Foundations for the Analysis of Surreal-valued genetic functions

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Outline

Two Motivating Questions

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1. Can surreal-valued genetically defined functions be uniformly bounded in terms of their complexity?

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2. Are there non-trivial examples found in the literature that correspond to a non-trivial ranking related to the bound provided for question 1?

Formal Languages (Terms)

- We let $\mathcal{L}$ denote a first-order language consisting of formal function symbols, formal relation symbols, and formal constant symbols.
- $\mathcal{L}$ terms are generated recursively from the set of constants in $\mathcal{L}$, variables x_i indexed by the natural numbers, and application of the function symbols of $\mathcal{L}$ applied to terms
- By abuse of notation $t := c|x|f(\bar{t})$
- $\mathcal{L}_r$ denotes the language of rings $\{+, -, \times\} \cup \{0, 1\}$. $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$.
- The set of $\mathcal{L}_r$ terms can be identified with $\mathbb{Z}[\bar{x}]$.

Formal Languages (Formula)

- Atomic formula are of the form $t_1 = t_2$ or $R(\bar{t})$.
- $\mathcal{L}$ formulas are the formula ϕ generated by the atomic formula, logical negation, conjunction, disjunction, and quantification
- By abuse of notation,
  $\phi := t_1 = t_2 | R(\bar{t}) | \neg\phi | \phi \vee \psi | \phi \wedge \psi | \exists x \phi | \forall x \phi$
- A type $p(\bar{v})$ is a set of $\mathcal{L}$ formula with free variables in $v_1, \ldots, v_n$.
- In $\mathcal{L}_{or}$, such that X is an ordered space satisfying the ring axioms, a cut C can be identified a partial type.
- Specifically, C are the partial types consisting of Left and Right formula, where the Left partial type consists of the atomic formula $t < x$, and similarly the Right partial type consists of the formula $x < t$, so that $L, R \subset X$ and $L < R$.

Formal Languages Examples

In $\mathcal{L}_{or}$,

- $v_1, v_2, v_3, \ldots$ are all terms
- $v_2 \cdot v_2 + v_3$ is a well-formed term
- $v_1 = v_2 \cdot v_2 + v_3$ is one atomic formula ϕ_1
- $v_3 < v_2$ is another atomic formula is another atomic formula ϕ_2
- $\varphi(v_1, v_2, v_3) = \phi_1 \wedge \phi_2$, is the formula that says,

$$v_1 = v_2 v_2 + v_3 \wedge v_3 < v_2$$

- $\psi = \forall v_1 \exists v_2 v_3 \varphi(v_1, v_2, v_3)$ is the sentence

Partizan Games and Disjunctive Game Compounds

- A combinatorial game G is a two player game, with players conventionally called Left and Right, who play alternately, and whose moves affect the position of the game according to rules.
- Games are partizan whenever these rules distinguish available moves to Left and Right players. Otherwise they are impartial
- If G and H are combinatorial games, H is a Left option of G whenever Left can move from G to H. Let L_G denote the set of all available direct moves for Left (and similarly R_G). Let G^L denote a generic Left move (and similarly G^R).
- We can form new games using the recursive disjunctive compound

$$G + H = \left\{G + H^L, G^L + H\right\} \Big| \left\{G + H^R, G^R + H\right\}.$$

- The class of Partizan games is an Abelian group; the subclass of numbers will be an ordered Abelian group.

Fundamental Examples

- (Endgame) $0 \equiv \{\} | \{\}$
- (Pos) $1 \equiv \{0\} | \{\}$
- (Neg) $-1 \equiv \{\} | \{0\}$
- (Fuz) $* \equiv \{0\} | \{0\}$

Outcome Classes and the Fundamental Theorem

There are four exhaustive outcome classes for all combinatorial games:

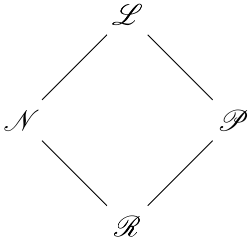
1. First player forces a win ($\mathcal{N}$)
2. Second player forces a win ($\mathcal{P}$)
3. Left player can always force a win ($\mathcal{L}$)
4. Right player can always force a win ($\mathcal{R}$)

Theorem (Fundamental Theorem)

If G is a Partizan game with normal play, then either Left can force a win playing first on G, or else Right can force a win playing second, but not both.

The Partial Ordering of Partizan Games

- By the Fundametal Theorem, there are exactly four equivalence classes to which a game belongs, which can be partially ordered according to the favorability of a game for the Left player:



- If $G, H \in \widetilde{PG}$, then $G \geq H$ if $o(G + X) \geq o(H + X)$ for all $X \in \widetilde{PG}$.
- $G = H$ if and only if $o(G - H) = \mathscr{P}$

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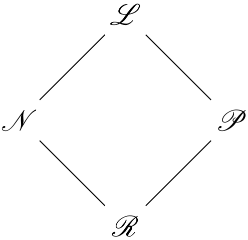


- If $G, H \in \widetilde{PG}$, then $G \geq H$ if $o(G + X) \geq o(H + X)$ for all $X \in \widetilde{PG}$.
- $G = H$ if and only if $o(G - H) = \mathscr{P}$ (G = 0 if and only if $o(G) = \mathscr{P}$)

The Partially Ordered Abelian Group of Partizan Games

- We denote the class of Partizan game values by $PG \equiv \widetilde{PG}/=$ where $=$ is the definable equality between Partizan games.
- The Class can be inductively constructed (e.g. hereditary property) as follows: for all ordinals α

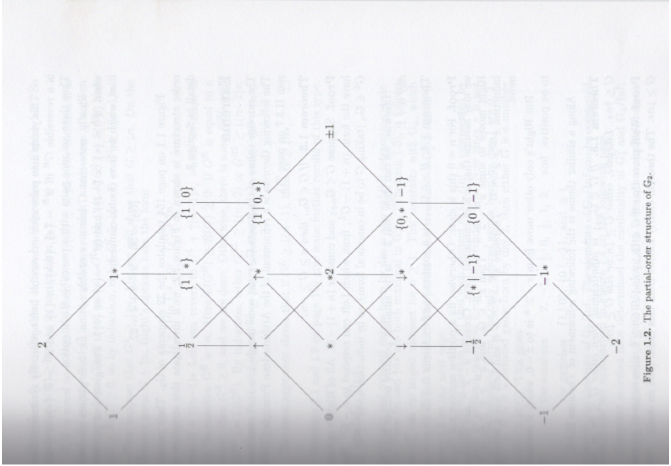
$$\widetilde{G_\alpha} = \{L_G|R_G : L_G, R_G \subseteq \bigcup_{\beta\in\alpha} \widetilde{G}_\beta\}$$

$$\widetilde{PG} = \bigcup_{\alpha\in\mathrm{On}} \widetilde{G}_\alpha$$

- The birthday of a partizan game G is the least ordinal α such that $G \in \widetilde{G}_\alpha$.
- We can define negation as follows

$$-G = \left\{-G^R\right\} \Big| \left\{-G^L\right\} = -R_G| - L_G = L_{-G}|R_{-G}.$$

Partial Ordering Up To Day 2 Game values



Numbers

- The game-value of G is a number whenever $L_H < R_H$ at every subposition H of G (numbers are games whose options are ordered sets of numbers)
- The canonical representation of a surreal number a is the positioned closed game $L_a | R_a$, such that $L_a < R_a$ and every $x \in L_a \cup R_a$ is simpler than a, i.e.

$$x <_s a \iff ((x < a \vee a < x) \wedge (L_x \subset L_a \wedge R_x \subset R_a)$$

$$\wedge (L_x \cup R_x \subsetneq L_a \cup R_a)$$

- The corresponding game tree is a full binary tree of height the Class of On.
- Numbers can be understood as unique minimal realization of cuts which correspond functions $\alpha \to 2$, and $a <_s b$ if for some $\beta \in \alpha$, $b \restriction \beta = a$.

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- We can build up a lexicographically, partially well-ordered binary tree such that each level is a canonical realization of the cuts of the level below (modulo details of the limit case).
- We can think of $|$ as a mapping from the Class of Conway cuts $\mathfrak{C}$ to the Class of the surreal numbers, sending cuts (F, G) to the unique minimal set-theoretic ranked number c such that $F < c < G$ and for all x such that $F < x < G$, we have $c \leq_s x$.

Two Key Results

Definition

Given pairs of sets (A, B) and (C, D), we say that (A, B) is **cofinal** in (C, D) if for all c in C and d in D there is an a in A and b in B such that $c \geq a$ and $d \leq b$.

Theorem (Gonshor Inverse Cofinality Theorem)

For all $a \in \text{No}$ if $a = F|G$ for a pair of sets (F, G), then the set pair (F, G) is cofinal in (L_a, R_a).

Theorem (Conway's Simplicity Theorem)

Let $L, R \subset \text{No}$ such that $L < R$ and $L \cup R \neq \text{No}$. Let $I = \{y \in \text{No} : L < y < R\}$. Then I is a non-empty convex class for which there exists a unique $x \in I$ such that $\iota(x) < \iota(y)$ for all $y \in I \backslash \{x\}$.

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- Numbers can be thought of as branches in a binary tree which describe a canonical type of Partizan game with an ordered structure.
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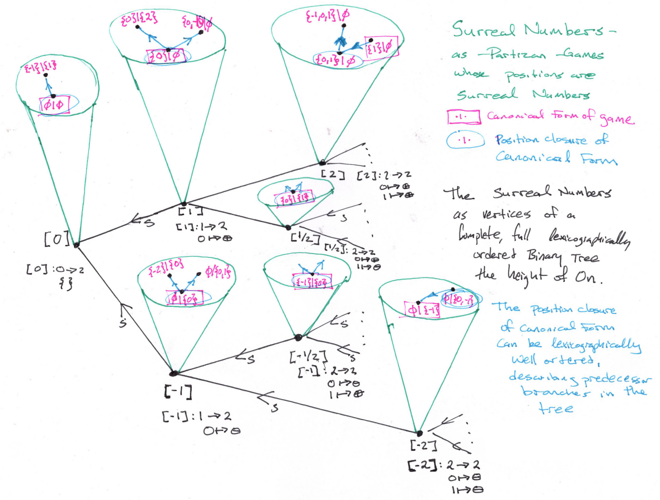
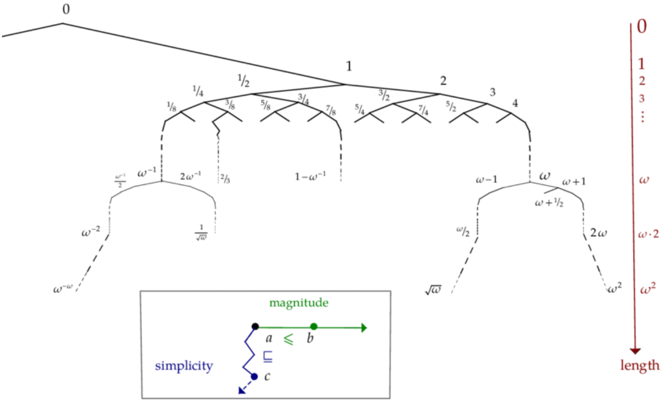
- We visualize this in terms of fibres modulo $=$ and up to restriction of minimal set theoretic rank over a surreal-number tree.
- This minimal set theoretic rank is what is meant by complexity in this talk. Specifically the length of a surreal number branch.

Numbers and Games



Above, numbers are vertices in a tree corresponding to the equivalence class of a recursively definable Conway cut.

Numbers and Simplicity



Length and the Weak Product Lemma

Let $\iota(a)$ denote the length of a surreal number, i.e. $\iota(a) = \alpha$ such that we can realize $a : \alpha \to 2$.
We do not know that

$$\iota(ab) \leq \iota(a)\iota(b)$$

in general.
Ehrlich-van den Dries established the following weak product inequality:

$$\iota(ab) \leq \omega[\iota(a)]^2[\iota(b)]^2.$$

Conway Normal Form and Sign sequences

- (Conway)No is a real-closed such that every element has a canonical normal form, $\sum_{\nu a} \omega^{y_i} r_i$, where (y_i) is a descending sequence of surreal numbers, and r_i is a non-zero real number.
- Every surreal number a has a corresponding sign sequence $(a) = \frown_{\phi a} \langle \alpha_i, \beta_i \rangle$, with $\iota(a) = \bigoplus(\alpha_i \oplus \beta_i)$
- Let a^+ denote the total number of $+$ appearing in the sign sequence of a, so

$$a^+ = \bigoplus_{\mu} \alpha_\mu$$

as an ordinal sum.

Sign Sequence Theorem 1

Theorem

Given $a = (\langle \alpha_i, \beta_i \rangle)_{i \in \phi a}$, *and for any* $\mu \in \phi a$, *we have*

$$\gamma_\mu := \bigoplus_{\lambda \le \mu} \alpha_\lambda,$$

then the sign sequence of (ω^a) *is given by*

$$(\omega^a) = \frown_{i \in \phi a} \langle \omega^{\gamma_i}, \omega^{\gamma_i + 1} \beta_i \rangle$$

Sign Sequence Theorem 2

Theorem

Given a positive real r with sign sequence $(\langle \rho_i, \sigma_i \rangle)$, the sign sequence of $\omega^a r$ is

$$(\omega^a) \frown \langle \omega^{a^+} \rho_0^\flat, \omega^{a^+} \sigma_0 \rangle \frown (\langle \omega^{a^+} \rho_i, \omega^{a^+} \sigma_i \rangle : 0 < i \leq \iota r)$$

with $\omega^{a^+} \rho$ and $\omega^{a^+} \sigma$ being the standard ordinal multiplication (with absorption). If r is a negative real, we reverse the signs.

Sign Sequence Reductions

- Given $a \in \mathrm{No}_{>0}$, define $a^\flat$ to be the surreal number attained by omitting the first $\oplus$ sign.
- Given $a \in \mathrm{No}_{<0}$, define $a^\sharp$ to be the surreal number attained by omitting the first $\ominus$ sign.
- Given a surreal number $a = \sum_{i \in \nu a} \omega^{a_i} r_i$ in normal form, we define the reduced sequence $(a_i^o | i \in \nu a)$ by omitting $\ominus$ from the following sign sequences:

- given $\gamma \in \mathrm{On}$, if $a_i(\gamma) = \ominus$ and there exists $j < i$ such that $a_j(\delta) = a_i(\delta)$ for all $\delta \leq \gamma$, then omit the δ^{th} $\ominus$;
- if i is a successor, $a_{i-1} \frown \ominus \sqsubset a_i$ and if r_{i-1} is not a dyadic rational, then omit $\ominus$ after a_{i-1} in a_i.

Sign Sequence Theorem 3

Theorem

Given $a = \sum_{i \in \nu a} \omega^{a_i} r_i$,

$$(a) = \frown_{i \in \nu a} (\omega^{a_i^o} r_i)$$

- B., '21 Classification of intervals of reduction

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- The length of a is an ordinal sum of the pair-wise ordinal sums $\alpha_i \oplus \beta_i$ (modulo a few details).
- A big obstruction to proving $\iota(ab) \leq \iota(a)\iota(b)$ are interactions between intervals where reduction takes place.

Motivating Example

- The disjunctive game compound $+$ is an order-preserving abelian group operation.
- We can recursively define a suitable notion of multiplication on the numbers by

$$ab = \{a^L b + ab^L - a^L b^L, a^R b + ab^R - a^R b^R\}|$$

$$\{a^L b + ab^R - a^L b^R, a^R b + ab^L - a^R b^L\}.$$

i.e.

$$(ab)^L := a^L b + ab^L - a^L b^L | a^R b + ab^R - a^R b^R$$
$$(ab)^R := a^L b + ab^R - a^L b^R | a^R b + ab^L - a^R b^L$$

- This is another example of a recursively definable function with the uniformity propery (a la Gonshor)

Adjoining New Function Symbols

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$$S(v, w) = \{c_1 + c_2 h(c_3 x + c_4) \colon c_1, c_2, c_3, c_4 \in K, h \in S\}.$$

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3. We then form Ring $R(v, w) := \text{No}[S(v, w)]_{P_S}$, where P_S is the cone of strictly positive polynomials with function from S.

Amendments to Rubinstein-Salzedo and Swaminathan

We want to choose sets $L_f(v, w)$ and $R_f(v, w)$ from $R(v, w)$ such that the order condition and cofinality condition will hold:

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  - (Cofinality Condition)

$$\forall x, y, z \in \text{No}((y < x < z) \rightarrow$$

$$L_f(y,z)[x] < f(x) < R_f(y,z)[x].$$

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  - Once f is defined over No, we prove that the cofinality condition holds, via (double) induction with respect to the natural sum of the lengths of the arguments and generation.

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Finally, set

$$f(x) := \{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^L(x) \colon f^L \in L_f(x^L, x^R)\}\}|$$

$$\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^R(x) \colon f^R \in R_f(x^L, x^R)\}\}$$

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- The base case of each function is always defined by the constants appearing in the Left and Right option sets, and the application of previously defined genetic functions at 0.
- When analyzing the complexity, we can induct on the complexity of our term sets and use pseudo-absolute values to bound our functions.

Example: exp and ω

- Let $[x]_n = \sum_{i \leq n} \frac{1}{i!} x^i$.
- Gonshor records Kruskals genetic definition of exp (p145 of Gonshor):

$$\exp(x) = \{0, \exp(x^L)[x - x^L]_n, \exp(x^R)[x - x^R]_{2n+1}\}$$

$$\{\frac{1}{[x^R - x]_n} \exp(x^R), \frac{1}{[x^L - x]_{2n+1}} \exp(x^L)\}$$

- Following Conway, we can define ω:

$$\omega(x) = \left\{0, n\omega(x^L)\right\} \Big| \left\{\frac{1}{2^n}\omega(x^R)\right\}$$

which agrees with the ordinal valued ω function when restricted to ordinals.

Gamma, Delta, and Epsilon Numbers

Following Hessenburg,

- γ ordinals additively indecomposable, i.e. for all $x, y \in \gamma$, $x \oplus y \in \gamma$.
- δ ordinals are additionally multiplicatively indecomposable, i.e. for all $x, y \in \delta$, $x \otimes y \in \delta$.
- Epsilon ordinals ϵ are ordinals ordinals $\ni 1$ such that for all $x \in \epsilon$, $x^\epsilon = \epsilon$.

$\Gamma\Delta E$ and Surreal-arithmetic

It has been known early on that closure under surreal-arithmetic operations corrsponds to truncating the surreal binary tree at heights corresponding to specific limit ordinals:

Gamma: No(λ) is an additive subgroup of No if and only if λ is a Gamma ordinal of the form ω^α for some ordinal α.

Delta: No(λ) is a commutative subring of No if and only if λ is a Delta ordinal of the form ω^γ for some Gamma ordinal γ.

Epsilon: No(λ) is a real-closed subfield of No if and only if λ is an Epsilon number.

My research builds off this to bound the complexity of structures defined in extension of $\mathcal{L}_{or}$ with sets of genetic symbols $\mathcal{G}$.

Pseudo-absolute values

Let $\varsigma : S_1 \to S_2$ be a map between two semi-rings. We say ς is a **pseudo-absolute value if the following holds:**

1. $\varsigma(x) = 0 \iff x = 0$;
2. $\varsigma(xy) \leq \varsigma(x)\varsigma(y)$;
3. $\varsigma(x + y) \leq \varsigma(x) + \varsigma(y)$

$\surd$

- (Cantor) Every ordinal α has a normal form $\sum_{i \in N\alpha} \omega\alpha_i$ such that $N\alpha \in \omega$ and $\alpha_i \geq \alpha_j$ for $i < j \in N\alpha$.
- For $\alpha_1, \alpha_2 \in \mathrm{On}$ with Cantor normal form $\sum_{j \in n_i} \omega^{\alpha_{i,j}}$ for $i = 1, 2$ we say $\alpha_1 \sim_\Gamma \alpha_2$ if and only if $\alpha_{1,0} = \alpha_{2,0}$.
- We define $\surd : \mathrm{On} \to \mathrm{On}$ by sending $\alpha \mapsto \omega^{\alpha_0}$.
- We extend this to $\mathrm{No} \to \mathrm{On}$ by precomposing with ι, i.e. $\surd : \mathrm{No} \to \mathrm{On}$ by $a \mapsto \surd(\iota(a))$.

Theorem (B.,'21)

$\surd$ *forms a pseudo-absolute value sending* No *to* $\{0, 1\} \cup \omega$"On, *i.e.*

- $\surd x = 0 \iff x = 0$;
- $\surd(x + y) \leq \surd x + \surd y$;
- $\surd(xy) \leq \surd x \surd y$.

Gonshor Fixed Point Theorem

Theorem (Gonshor Fixed Point)

Suppose $f : \mathrm{No} \to \mathrm{No}$ *satisfies the following properties:*

1. *For all* $a \in \mathrm{No}$, $f(a)$ *is a power of* ω;
2. $a < b \Rightarrow f(a) < f(b)$;
3. *There are two fixed sets* C *and* D *such that whenever* $a = G|H$ *where* G *contains no maximum and* H *contains no minimum, then* $f(a) = (C \cup f(G))|(D \cup f(H))$.

Then the function g *defined by*

$$g(b) := \left\{ f^{(n)}(C), f^{(n)}(2g(b^L)) \right\} \Big| \left\{ f^{(n)}(D), f^{(n)}(\frac{1}{2}g(b^R)) \right\}$$

is onto the set of all fixed points of f *and satisfies the above hypotheses with respect to the sets* $f^{(n)}(C)$ *and* $f^{(n)}(D)$, *where* $f^{(n)}$ *denotes the* n^{th} *iterate of* f. *Furthermore, there is a* On*-length family of functions* f_α *satisfying all three conditions, such that* $f_0 = f$ *and for* $\alpha > 0$, f_α *is onto the set of all common fixed points of* f_β *for* $\beta \in \alpha$ *and satisfies condition (iii) with respect to the sets* $h(C)$ *and* $h(D)$ *where* h *runs through all finite compositions of* f_β *for* $\beta \in \alpha$.

Veblen hierarchy

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- A normal ordinal valued function φ_0 is any continuous (with respect to the order topology) strictly increasing ordinal valued function.

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- Given a normal function φ_0, the Veblen functions with respect to φ_0 are the sequence of functions $\langle \varphi_\alpha \colon \alpha \in \mathsf{On} \rangle$ such that each φ_α enumerates the common fixed points of φ_β for every $\beta \in \alpha$.

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- The Veblen hierarchy is the Class of functions $\langle \varphi_\alpha : \alpha \in \mathrm{On} \rangle$ generated by $\varphi_0(x) = \omega^x$.
- Finally, we have the following ordering on the Veblen hierarchy:

$$\varphi_\alpha(\beta) < \varphi_\gamma(\delta) \iff$$

$$(\alpha = \gamma \wedge \beta < \delta) \vee (\alpha < \gamma \wedge \beta < \varphi_\gamma(\delta)) \vee (\alpha > \gamma \wedge \varphi_\alpha(\beta) < \delta))$$

Veblen hierarchy

Recall

$$\omega(x) = \left\{0, \omega(x^L)n\right\} \Big| \left\{\omega(x^R)2^{-n}\right\}$$

and

$$\epsilon(x) = \left\{0, \omega^{(n)}(0), \omega^{(n)}(\epsilon(x^L)+1)\right\} \Big| \left\{\omega^{(n)}(\epsilon(x^R)-1)\right\}$$

- Because ω is a genetic function, it is immediate that every Veblen function is a genetic function
- In fact, we could show that the construction of g in GFPT given $\varphi_0(x) = \omega(x)$ is equicofinal with the construction of $\epsilon(x)$.
- Our primary motivation here is to identify for every $g \in \mathscr{G}$, the least α such that for all $\gamma \in \mathrm{On}$, if $x \in \mathrm{No}(\gamma)$ then $g(x) \in \mathrm{No}(\varphi_\alpha(\gamma))$.

Partial Veblen Rank

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Partial Veblen Rank

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- $VR(g, \gamma) \geq 0$;
- $VR(g, \gamma) \geq \lambda$ for limit ordinals λ if and only if $VR(g, \gamma) \geq \beta$ for all $\beta \in \lambda$;
- $VR(g, \gamma) \geq \alpha + 1$ if and only if there is an $x \in \mathrm{No}(\epsilon_\gamma)$ such that $\sqrt{g(x)} \geq \varphi_{\alpha+1}(\gamma)$.

Full Veblen Rank

Full Veblen Rank

- We say $VR(g, \gamma) = \alpha$ whenever $VR(g, \gamma) \geq \alpha$ and $VR(g, \gamma) \not\geq \alpha + 1$, i.e. α is the least ordinal such that for all $x \in \text{No}(\varphi_1(\gamma))$, $g(x) \in \text{No}(\varphi_{\alpha+1}(\gamma))$.

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- We then define the Veblen rank of g by $VR(g):=\bigcup_{\gamma\in\mathbf{On}} VR(g,\gamma)$.
- We can extend this definition to $g:\mathrm{No}^n\to\mathrm{No}$ by noting that $\iota(\bar{x})$ is the Hessenberg sum of the lengths of the components, so we can interpret $\mathrm{No}^n(\epsilon_\gamma)$ as the initial subset of No^n consisting of n-tuples of branches whose Hessenberg sum is less than ϵ_γ.

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- Whenever $VR(g) \geq \alpha$ for all $\alpha \in \mathrm{On}$, rather than denote this by saying the rank is ∞, we indicate this by saying the rank is On.

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- We say $VR(g,\gamma) = \alpha$ whenever $VR(g,\gamma) \geq \alpha$ and $VR(g,\gamma) \not\geq \alpha + 1$, i.e. α is the least ordinal such that for all $x \in \mathrm{No}(\varphi_1(\gamma))$, $g(x) \in \mathrm{No}(\varphi_{\alpha+1}(\gamma))$.
- We then define the Veblen rank of g by $VR(g) := \bigcup_{\gamma \in \mathrm{On}} VR(g,\gamma)$.
- We can extend this definition to $g : \mathrm{No}^n \to \mathrm{No}$ by noting that $\iota(\bar{x})$ is the Hessenberg sum of the lengths of the components, so we can interpret $\mathrm{No}^n(\epsilon_\gamma)$ as the initial subset of No^n consisting of n-tuples of branches whose Hessenberg sum is less than ϵ_γ.
- Whenever $VR(g) \geq \alpha$ for all $\alpha \in \mathrm{On}$, rather than denote this by saying the rank is ∞, we indicate this by saying the rank is On.
- As an aside, we can extend the notion of Veblen rank from entire genetic functions, to those that are defined on convex intervals of surreal numbers, like all positive surreal numbers in the case of log.

Main Theorem

Theorem (Main Theorem (B., '21))

Every genetic function g has Veblen rank in On, *i.e.* $\exists\gamma \in \mathrm{On}\forall\beta \in \mathrm{On}(\gamma \in \beta \Rightarrow VR(g,\beta) \leq VR(g,\gamma))$.

This proof is built on the following:

Lemma (B, '21)

For all surreal-valued genetic functions f, g,

- $VR(f+g) \leq \max\{VR(f), VR(g)\}$
- $VR(fg) \leq \max\{VR(f), VR(g)\}$
- $VR(f \circ g) \leq \max\{VR(f), VR(g)\}$
- *For a set* S *of genetic functions, and any term* t *generated by* $\mathcal{L}_{oring} \cup S$,

$$VR(t^n) \leq VR(t) \leq \sup\{VR(g) \colon g \in S\}$$

Summary of Proof

The most important piece:

Theorem (B, '21)

Suppose that f is a genetic function whose Left and Right options sets has order type τ, i.e. o.t.$|L_f \cup R_f| = \tau$, and $L_f \cup R_f$ consists of genetic functions g_i indexed by some set I, such that for each $i \in I$, $VR(g_i) = \alpha_i$. Set $\alpha = \sup_I \alpha_i$, and $\mu = \max\{\tau, \alpha\}$. Then $VR(f) \leq \mu + 1$.
Further, $VR(f) = \mu + 1$ if and only if for at least one $\gamma \in \mathrm{On}$, there is some $x_\gamma \in \mathrm{No}(\epsilon_\gamma)$ for which there is an infinite enumeration K of terms in $L_f \cup R_f$, such that for $k, k' \in K$, $\varphi_k(\gamma) \leq \sqrt{t_k}(x_\gamma) \leq \sqrt{t_{k'}}(x_\gamma)$ when $k < k'$ and such the sequence $(\sqrt{t_k}(x_\gamma))$ is cofinal with $\varphi_{\mu+1}(\gamma)$.

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Further, $VR(f) = \mu + 1$ if and only if for at least one $\gamma \in \mathrm{On}$, there is some $x_\gamma \in \mathrm{No}(\epsilon_\gamma)$ for which there is an infinite enumeration K of terms in $L_f \cup R_f$, such that for $k, k' \in K$, $\varphi_k(\gamma) \leq \sqrt{t_k}(x_\gamma) \leq \sqrt{t_{k'}}(x_\gamma)$ when $k < k'$ and such the sequence $(\sqrt{t_k}(x_\gamma))$ is cofinal with $\varphi_{\mu+1}(\gamma)$.

After proving the above theorem, we induct on complexity of genetic functions to prove Main Theorem 1.

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- Throughout these proofs we make frequent use of the fact that numbers, and the value of genetic functions, is defined with respect to the minimal set theoretic realization satisfying $L < c < R$ for the corresponding Left and Right options.

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- Throughout these proofs we make frequent use of the fact that numbers, and the value of genetic functions, is defined with respect to the minimal set theoretic realization satisfying $L < c < R$ for the corresponding Left and Right options.
- As a consequence, the complexity of arguments can only grow so rapidly, determined entirely by the complexity of constants, the respective order type of the Left and Right option sets, and the complexity of functions of earlier generations.

Genetic Functions are strongly tame

A function $f : \mathrm{No}^{n+1} \to \mathrm{No}$ is **strongly tame** if and only if for all $a < b \in \mathrm{No}$, $\bar{e} \in \mathrm{No}^n$, $d \in \mathrm{No}$, either $f(x, \bar{e})$ is constant or there exists $\zeta_0, \ldots, \zeta_m \in \mathrm{No}^{\mathcal{D}}$ such that $a = \zeta_0 < \ldots < \zeta_m = b$ and for i =0,...,m-1

$$\forall x \in (\zeta_i, \zeta_{i+1}), f(x, \bar{e}) > d \text{ or}$$

$$\forall x \in (\zeta_i, \zeta_{i+1}), f(x, \bar{e}) < d$$

As a direct consequence of the Main Theorem, we also can see that once we know a genetic function has a given Veblen rank, then we can always bind that genetic function, i.e. genetic functions are strongly tame.

Examples

The following have zero Veblen rank:

- Identity
- Addition
- Negation
- Multiplication
- exp
- ω
- log

Additionally, each Veblen function $\varphi_\alpha(x)$ has Veblen rank α (B, '21)

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$$\kappa(x) := \left\{\exp^{(n)}(0), \exp^{(n)}(\kappa(x^L))\right\} \Big| \left\{\log^{(n)}(\kappa(x^R))\right\}$$

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- One can check that $\kappa(1) = \epsilon_0 = \varphi_1(0)$, and so we have a witness to $VR(\kappa) \geq 1$.

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- One checks by induction that $VR(\kappa, \gamma) \leq 1$ on all γ, while using some bounds on the complexity of exp and log established by van den Dries and Ehrlich.

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- One can check that $\kappa(1) = \epsilon_0 = \varphi_1(0)$, and so we have a witness to $VR(\kappa) \geq 1$.
- One checks by induction that $VR(\kappa, \gamma) \leq 1$ on all γ, while using some bounds on the complexity of exp and log established by van den Dries and Ehrlich.
- In particular, one checks for all $x \in \mathrm{No}(\epsilon_\gamma)$ that $\sqrt{(\kappa(x))} < \varphi_2(\gamma)$, which follows using the aforementioned inequalities to show that

$$\sqrt{(\kappa(x))} \leq \varphi_1(\iota x) < \varphi_1(\varphi_1(\gamma)) < \varphi_2(\gamma)$$

$VR(\lambda) = 1$ (B., '21)

Definition

Let $a \in \mathrm{No}^{>0}_{>0}$, i.e. a is a positive infinite surreal number. We say a is **log-atomic** if for all $n \in \mathbb{N}$, there is a $b_n \in \mathrm{No}$ such that for the n-fold iterate of log we have

$$\log^{(n)}(a) = \omega^{b_n}.$$

We denote the class of log-atomic numbers by $\mathbb{L}$.

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- Then defining

$$\lambda(x) = \left\{ m, \exp^{(n)}(n \cdot \log^{(n)} \lambda(x^L) \right\} \Big| \left\{ \exp^{(n)} \left(\frac{1}{m} \log^{(n)} \lambda(x^R) \right) \right\}$$

one sees the λ numbers correspond to the log-atomic numbers

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one sees the λ numbers correspond to the log-atomic numbers
- following applications of the aforementioned inequalities in the previous slide, we also establish that $VR(\lambda) = 1$.

Questions and Future directions

- Can similar work be done for characteristic p cases? p-adic cases?
- Generalizing work on homogeneous and model-complete theories? (when are we guaranteed to get initial embeddings)
- What about realization in *exotic* set theories?

Thank You

THANK YOU!