Cohesive Homotopy Type Theory: A Gentle Introduction To The World From The Perspective Higher Topos Theory

Alexander Berenbeim

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Overview: Why Bother With CHoTT?

- Motivating Mathematics Is Always A Tricky Proposition.
- Cohesive Types Are Like The Types We Are Familiar With, Just With More Structure
- Interpreting Type Theory Internal To Categories With Geometric Structures
- Goal: To understand how *cohesion* arises out of a quadruple of adjoint functors which give rise to a triple of adjoint modalities.

Preliminaries: Categories

Definition

A **category** A consists of :

1. A type A_0 of **objects**. If a is an object of A_0, we may denote this by $a : A$;
2. for each $a, b : A$, a set $\mathtt{Hom}_A(a, b)$ of **morphisms** $f : a \to b$;
3. for each $a : A$, a morphism $1_a : \mathtt{Hom}_A(a, a)$;
4. for each $a, b, c : A$, a function $\mathtt{Hom}_A(b, c) \to \mathtt{Hom}_A(a, b) \to \mathtt{Hom}_A(a, c)$ denoted by $g \circ f$;
5. for each $a, b : A$ and $f : \mathtt{Hom}_A(a, b)$, $f = 1_b \circ f$ and $f = f \circ 1_a$.
6. for each $a, b, c, d : A$ and $f : \mathtt{Hom}_A(a, b), g : \mathtt{Hom}_A(b, c)$, and $h : \mathtt{Hom}_A(c, d)$, $h \circ (g \circ f) = (h \circ g) \circ f$

Preliminaries: Functors

Definition

Let A, B be categories. A **functor** $F : A \to B$ consists of:

1. A function $F_0 : A_0 \to B_0$, denoted by F;
2. for each $a, b : A$, a function $F_{a,b} : \mathtt{Hom}_A(a, b) \to \mathtt{Hom}_B(F(a), F(b))$, also generally denoted by F;
3. for each $a : A$, $F(1_a) = 1_{F(a)}$;
4. for each $a, b, c : A, f : \mathtt{Hom}_A(a, b), g : \mathtt{Hom}_A(b, c), F(g \circ f) = Fg \circ Ff$.

Preliminaries: Natural Transformations

- If $F, G : A \to B$ are functors, then a **natural transformation** $\alpha : F \to G$ consists of
  1. (components) for each $a : A$, $\alpha_a : \mathtt{Hom}_B(Fa, Ga)$
  2. (naturality) for each $a, b : A$ and $f : \mathtt{Hom}_A(a, b)$, $Gf \circ \alpha_a = \alpha_b \circ Ff$
- Functors which preserve finite limits are **left exact** and dually, functors which preserve finite co-limits are **right exact**.

"Adjoint functors arise everywhere"

- We say two functors F, G are **adjoint** if for all $a : A, b : B, \mathtt{Hom}_A(a, Gb) \cong \mathtt{Hom}_B(Fa, b)$.
- We denote this by $F \dashv G$.
- Adjoints arise when there are natural transformations $\eta : 1_A \to GF$ and $\varepsilon : FG \to 1_B$ which satisfy the zig-zig identities: $(\varepsilon F)(F\eta) = 1_F$ and $(G\varepsilon)(\eta G) = 1_G$.
- Let $F \dashv G$. Then F preserves all colimits of A and G preserves all limits of D. If F is a left exact functor, then right adjoint G preserves all finite limits and co-limits. Hence F is exact.

The World Of Adjoints

- Definition
  A pointed object classifying monomorphisms is a **subobject classifier**
- Example
  Univalence implies that the type $\texttt{Prop} :\equiv \sum_{X:\mathcal{U}} \texttt{isProp}(X)$ classifies monomorphisms.
- Definition
  Let A, B be categories which have finite limits, are cartesian closed and have a **subobject classifier**, and let $f : A \to B$. We say f is a **geometric morphism** if there is a pair of functors (f^*, f_*) of the form $A \underset{f_*}{\overset{f^*}{\overset{\leftarrow}{\rightarrow}}} B$ such that f^* is left exact and $f^* \dashv f_*$.

1-Topos and ...

- Definition

  In the previous slide, we introduced the notion of geometric morphisms. These are morphisms over a very special kind of category called a **topos**. Specifically, a topos is a category $\mathcal{E}$ which

  1. has finite limits
  2. has an object $\Omega : \mathcal{E}$, called the **subobject classifier**, with a function P which assigns to each object $a : \mathcal{E}$ an object $Pa : \mathcal{E}$, where Pa is called the **power object of a**;
  3. the functors $\mathrm{Sub}_{\mathcal{E}}$ and $\mathtt{Hom}_{\mathcal{E}}(b \times -, \Omega)$ such that for each object $a : \mathcal{E}$, we have two natural isomorphisms $\mathrm{Sub}_{\mathcal{E}} a \cong \mathtt{Hom}_{\mathcal{E}}(a, \Omega)$ and $\mathtt{Hom}_{\mathcal{E}}(b \times a, \Omega) \cong \mathtt{Hom}_{\mathcal{E}}(a, Pb)$.

- Example

  The canonical example of a topos is the category of sets, where the subobject classifier consists of the characteristic functions and $\Omega = \{0, 1\}$.

… Beyond

Definition

A **local geometric morphism** is an adjoint triple $f^* \dashv f_* \dashv f^! : B \to A$ such that for all $a, b : B$, f^* is such that

1. (full) $f^* : \mathtt{Hom}_B(a, b) \twoheadrightarrow \mathtt{Hom}_A(f^*a, f^*b)$
2. (faithful) $f^* : \mathtt{Hom}_B(a, b) \rightarrowtail \mathtt{Hom}_A(f^*a, f^*b)$.

Definition

A **local topos** $\mathcal{E}$ is a sheaf topos where the global section geometric morphism $\mathcal{E} \underset{\Gamma}{\overset{Lconst}{\leftrightarrows}} \mathtt{Set}$ has a further right adjoint $\mathtt{coDisc} : \mathtt{Set} \hookrightarrow \mathcal{E}$, i.e. $Lconst \dashv \Gamma \dashv \mathtt{coDisc}$.

Some Examples of Adjoints

- Example

  Consider the unit type $\mathbf{1}$ and an arbitrary category A. Clearly, $G : A \to \mathbf{1}$ is a unique functor. If $F \dashv G$, then for any $a : A$, we find $\mathtt{Hom}_A(F(\star), a) \cong \mathtt{Hom}_{\mathbf{1}}(\star, G(a))$, since only one map exists from $\star \to G(a)$

- Example

  Let A=$\mathtt{Top}$, the category of topological spaces. Now consider functors $F, G : \mathtt{Set} \to \mathtt{Top}$, which takes a set to its discrete and indiscrete topologies respectively. We find that $F \dashv U \dashv G$, which is an example of an **adjoint triple**.

Monads In Categories

- A **monad** in a category A is a triple given by
  - an endofunctor T
  - a natural transformation $\eta : 1_A \to T$ called the **unit of T**
  - a natural transformation $\mu : T \circ T \to T$ called the **multiplication**.
  - These natural transformations satisfy:

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow & \downarrow \mu & \swarrow & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ T\mu \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

- A **comonad** on a category A is a monad on its dual category A^{op}.

Modalities

- Roughly speaking, a modality is a function $M : U \to \texttt{Prop}$ that tells us for every type A whether A has a given property M.
- If M is a modality, then for every type A, there is anther type $\bigcirc(A)$ such that $M(\bigcirc(A))$holds.

Example

$\texttt{isSet}{:}U \to \texttt{Prop}$ is a modality for which the $\bigcirc$ is given by the set truncation $\|A\|_0$.
In HoTT, the most frequently encountered modalities are the $n-$truncations.

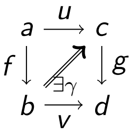
Modalities as Stable Factorization Systems

We can think of modalities as **stable factorization systems**. That is

Definition

Let A be a category. Let (E, M) form two classes of morphisms. If (E, M) form two classes in A such that

1. for every $f : \mathtt{Hom}_A(a, b)$ factors into $f = r \circ l$, with $l : E$ and $r : M$ such that these factorizations are unique up to isomorphism;
2. E, M contain all isomorphisms;
3. and are closed under composition;
4. they satisfy the lifting problem:



The Heart of the Matter: Adjoint Monads

- Definition

  An **adjoint cylinder** is an adjoint triple $F \dashv G \dashv H$ such that the adjoint pair on of the two sides consists of identity functors and the other side consists of an idempotent monad or comonad.

  - Every adjoint triple induces an adjoint pair of endofunctors that underlie a monad induced by adjunction.

What Is Cohesion Anyway? (Hint: Adjoint Triples)

- Some familiar cohesive structures: open balls in topological spaces or smooth structures.
- Any type admits both discrete cohesion where no distinct points cohere non-trivially, and a codiscrete cohesion, where all points cohere in every possible way admitted by the cohesive structure.
- Broadly speaking, cohesion is an adjoint triple of modalities

$$\texttt{modality} \dashv \texttt{comodality} \dashv \texttt{modality} \equiv \int \dashv \flat \dashv \sharp$$

Coming to Terms With Cohesion

- Definition
  $\sharp :\equiv \texttt{coDisc} \circ \Gamma$, where the codiscrete objects are the modal types.
- Definition
  $\flat :\equiv \texttt{Disc} \circ \Gamma$, where the discrete objects are the modal types.
- Definition
  $\int :\equiv \texttt{Disc} \circ \Pi$, where the "shape" modality $\int$ builds out of an additional left adjoint Π, which preserves finite products.
  - So we identify the adjoint 4-tuple of functors with the adjoint triple of modalities:

$$\Pi \dashv \texttt{Disc} \dashv \Gamma \dashv \texttt{coDisc} \equiv \int \dashv \flat \dashv \sharp$$

Example:Mengen/Kardinalen (Quantity), Continuum (Infinitesimally Cohesive), and Cohesive Sets

- The motivating example: Lawvere's analysis of Cantor's account of *Mengen* and *Kardinalen*.
- The notion of **quantity** is an adjoint between discreteness and continuity given by $\flat \dashv \sharp$.
- The geometric notion of continuum geometry with the adjoint cylinder from $\int \dashv \flat$ and the natural transformation $\flat X \to X \to \int X$.
- If this transformation is an equivalence, ie $\flat \xrightarrow{\simeq} \int$, then $\mathbf{H}$ is **infinitesimally cohesive**, in the sense that objects are built from precisely one point in each cohesive piece.
- A **cohesive set** is an adjoint triple of these modalities $\int \dashv \flat \dashv \sharp$.

Example: Reflexive Graphs (Thanks Tobias!)

- Let $\mathtt{RGr}$ be the category of reflexive graphs.
- Let $\Gamma : \mathtt{RGr} \to \mathtt{Set}$, defined by taking a graph to the set of its vertices.
- How to characterize the adjoint quadruple $\Pi \dashv \mathtt{Disc} \dashv \Gamma \dashv \mathtt{coDisc}$?
  - $\mathtt{coDisc}$ should be a functor from $\mathtt{Set}$ to $\mathtt{RGr}$ which *completely* coheres.
  - Disc should be a functor from $\mathtt{Set}$ to $\mathtt{RGr}$ which *completely* repulses.
  - Π is a functor that should send each reflexive graph to a set of its path components.

Questions?