Introduction to Partizan Games and the Surreal Numbers

Alexander Berenbeim

Working In NBG

Combinatorial Games

Combinatorial Games and Disjunctive Compounds

Partizan Games Form A Partially Ordered Abelian Group

Universal Homogeneous Models

PG as a Universal Embedding Object

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Aug 1st, 2021

Outline

- Working In NBG

1. Combinatorial Games
   - Combinatorial Games and Disjunctive Compounds
   - Partizan Games Form A Partially Ordered Abelian Group

2. Universal Homogeneous Models
   - PG as a Universal Embedding Object

Main Ideas for Three Talks

- Talk 1: Induction Principles and NBG
- Talk 1: The Partially Ordered Abelian Group Structure of Partizan Games
- (Talk 1?): The Embedding Theorem for Partizan Games
- Talk 2: Defining Surreal-Valued Genetic Functions
- (Talk 2?): Analysis of Surreal-valued Genetic Functions
- Defense: Ranking the complexity of genetic functions
- Defense: Theories $T_{\mathcal{G}}$ extending RCF.

Key Terms Ideas For This Talk

- Combinatorial game, game position, Left/Right options
- rule set, input complexity, solution
- normal play, misere play
- game outcome, $\leq, +$
- dominated options, reversible options, canonical form
- simplicity

Formalization of Induction Principles

- [Conway Induction Principle] "If P is some proposition that holds for x whenever it holds for all x^L, x^R, then P holds universally
- Counterpart:Axiom of Restriction/Foundation in ZF, "If P is a proposition that holds of set x whenever it holds for all $y \in x$, then P holds for every set"
- Anticipated Issue: Definable Equivalence Classes will be proper Classes, so it cannot be an element of another Class
- Conway Workaround: Definable equivalence classes E are defined with respect to minimal nonempty intersection with $E \cap V_\alpha$
- Ehrlich Workaround: Explicitly work with Complete Binary Tree Height Ordinals (for surreal numbers only)

Why NBG?

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- NBG is a conservative extension of ZFC

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- Class comprehension scheme where we can quantify over sets (but not all Classes).
- The Class comprehension scheme is as follows, given wff ϕ,

$$\forall \vec{y} \exists Z \forall x (x \in Z \iff \phi(x, \bar{y}))$$

NBG Class Comprehension Scheme

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3. (Axiom of Complement)

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4. (Axiom of Domain)$\forall X \exists Y \forall u(u \in Y \leftrightarrow \exists v(\langle u, v\rangle \in X))$
5. $\forall X \exists Z \forall u \forall v(\langle u, v\rangle \in Z \leftrightarrow u \in X)$
6. $\forall X \exists Z \forall u \forall v \forall w(\langle u, v, w\rangle \in Z \leftrightarrow \langle v, w, u\rangle \in X)$
7. $\forall X \exists Z \forall u \forall v \forall w(\langle u, v, w\rangle \in Z \leftrightarrow \langle u, w, v\rangle \in X)$

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2. and the effects of each move are fully determined beforehand by some ruleset Γ that describe how players are to move with respect to their available options, which we define below, given the games' current position.

We describe such games as containing no hidden information, and no chance elements respectively.

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  1. Γ is the partial mapping $G \mapsto \langle L, R \rangle$, where L, R are sets of games such that Γ is defined for each game.
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  4. $N : dom(\Gamma) \to \mathbb{N}$. $N(G)$ is the input complexity of a position G.

Representing Combinatorial Games

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- This is in contrast to the more common $G = \{L|R\}$ notation, which conflicts with notation for set/class comprehension.

Describing Combinatorial Games (Constraints and Conventions)

- Constraint A game is impartial if both players have the same moves available to them at each subposition of G. Otherwise, a game is partizan if each player has a distinct move set.
- Constraint A game is loopfree if all runs are finite-length; otherwise loopy.
- Constraint A game is finite if there are finitely many subpositions; otherwise transfinite.
- Convention A game is in normal play if the last player (previous) to move wins, with the convention that the game is over when at least one of the players has no move available moves (i.e. an empty Option set).
- We say a game is over whenver the option set is empty for the player who is currently moving.

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RECALL: For two combinatorial games G and H, we say H is a Left option (respectively Right) of G if Left can move directly from G to H.

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RECALL: For two combinatorial games G and H, we say H is a Left option (respectively Right) of G if Left can move directly from G to H.We say H is a subposition of G if there exists a sequence of consecutive moves leading from G to H.
NOTATION: We indicate a left option of G by G^L and a right option of G by G^R, while the set of left options is denoted by L_G and right options by R_G.

Disjunctive Compounds

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- For two combinatorial games, G and H, the game represented by G + H means that the players move in exactly one of the two component games

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Disjunctive Compounds

- For two combinatorial games, G and H, the game represented by G + H means that the players move in exactly one of the two component games
- The range of options are displayed formally as

$$G + H \equiv \left\{ G^L + H, G + H^L \right\} \Big| \left\{ G^R + H, G + H^R \right\}$$

Example 2: Dawson's Kayles

Players are given rows of boxes and on their turn remove exactly two adjacent boxes with the normal play convention that we stop when current player cannot remove two adjacent boxes.

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- Supposing Left moves first and plays/changes the position to option K_7. K_7 becomes the current game position, and the available options for the Right (and Left player) are updated to

$$K_5, K_4, K_2 + K_3$$

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So Right has to choose two adjacent boxes from

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The disjunctive sums of the game allow us to reduce the study of the outcome of an arbitrary position to understanding the structure of individual strips (or heaps in the case of `Nim`).

Identity and Classifying Games Part 1

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- The four outcome Classes are:

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  4. Right can force a win no matter who moves first, denoted by $\mathscr{R}$.

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  4. Right can force a win no matter who moves first, denoted by $\mathscr{R}$.
- A solution to (Γ, N) efficiently computes $o(G)$ for every

The "Fundamental" Equivalence

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- We consider three parameters when defining equality of games:
  1. the definition of outcome, $o(G)$;
  2. the definition of *sum* (or more appropriately, the compound of games) $G + H$;
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- The Fundamental Equivalence is given by the form $G = H$ if $o(G + X) = o(H + X)$ for all X, i.e. invariant under translation by the class of games.
- Consequently, the zero position is a game $G = 0$, such that $o(G + X) = o(X)$ for all X.

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- A combinatorial game is impartial, if Left and Right always have exactly the same moves available from every subposition, like in Dawson's Kayles.

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- A combinatorial game is impartial, if Left and Right always have exactly the same moves available from every subposition, like in Dawson's Kayles.
- A combinatorial game is partizan if it is not necessarily impartial

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- A combinatorial game is impartial, if Left and Right always have exactly the same moves available from every subposition, like in Dawson's Kayles.
- A combinatorial game is partizan if it is not necessarily impartial (this Class does include impartial games)
- The class can be inductively constructed (e.g. hereditary property) as follows: for all ordinals α

$$\widetilde{G_\alpha} = \{L_G | R_G : L_G, R_G \subseteq \bigcup_{\beta \in \alpha} \widetilde{G_\beta}\}$$

$$\widetilde{PG} = \bigcup_{\alpha \in \mathrm{On}} \widetilde{G_\alpha}$$

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- The birthday of a partizan game G is the least ordinal α such that $G \in \widetilde{G}_\alpha$.

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- We can partially order $\widetilde{PG}$ with respect to outcome class (or equivalently, we can recursively define a partial ordering which we use to define the Fundamental equivalence $=$).

The Fundamental Theorem

Theorem (Fundamental Theorem)

If G is a Partizan game with normal play, then either Left can force a win playing first on G, or else Right can force a win playing second, but not both.

Proof.

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Proof.

By induction and symmetry, it suffices to consider a Left option $G^L \in L_G$.

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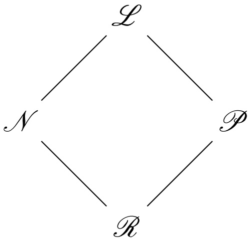

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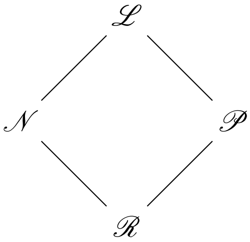

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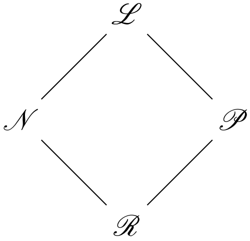

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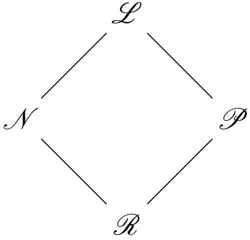

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$G \geq H$ if and only if $o(G+X) \geq o(H+X)$ for all $X \in PG$ if and only if $o(G-H) \geq \mathscr{P}$

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Example: some simple Games and their ordering

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$$0 < \uparrow < \uparrow + \uparrow < \uparrow + \uparrow + \uparrow < \cdots$$

Example: Dicotic games

Consider the games

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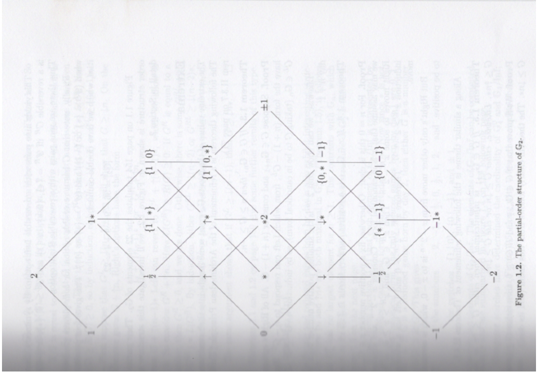
Partial Ordering Up To Day 2 Game values



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- The restriction of $\geq$ to No is equivalent to the lexicographical ordering defined by $- < \emptyset < +$ with $2 \equiv \{-,+\}$.

Brief Intermission

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- Norton multiplication of two games is given by the recursive construction:

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One (Some) Approach to Multiplication:

- Norton multiplication of two games is given by the recursive construction:

$$G.H = \begin{cases} \sum_G H & G \in \mathbb{Z} \\ \{G^L.H + H^L, G^L.H + (H + (H - H^R))\}| & \\ \{G^R.H - U^L, G^R - (H + (H - H^R))\} & ow. \end{cases}$$

- For subgroups $\mathbb{Z} \subseteq A \subset$ No, and games $G, H \in A$

$$(G + H).U = G.U + H.U$$

and

$$G \geq H \Rightarrow G.U \geq H.U$$

for all games $U \in PG$.

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Some Approach to Multiplication:

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- This doesn't have other desirable properties of multiplication, nor is overheating well-defined modulo $=$ for all partizan games (in fact, no global definition of multiplication that is well defined for PG has been found).

Examples: Order n subgroups of PG and the structure of the parially ordered abelian group

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- For all $n \in \mathbb{Z}^+$, the $G = \{2\} | \{-1, \{0\} | \{-4\}\}$ is such that $n.G || 0$;
- All submonoids S of $\mathbb{Z}_{\geq 0}$ are finitely generated

Back to the ur-motivation

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- The class of Partizan games forms a partially ordered abelian group

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- Provided one admits a strong version of choice (such as the existence of a well-ordering of the universe), by a back-and-forth argument, this theorem characterizes *PG* up to isomorphism.
- The theorem is *not true* if we restrict to the class of short (i.e. games with finitely many options).

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- If S is a set of games, S is hereditary if for every $x \in S$, all $L_x, R_x \subset S$.

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- It may be fruitful to develop Lurie's notion of hereditary sets and the h-hierarchy in ways similar to the s-hierarchy
- If S is a set of games, S is hereditary if for every $x \in S$, all $L_x, R_x \subset S$. (We can enlarge S to a hereditary set).

Lurie's Proof (Rough Outline)

1. Weak Embedding
2. Construction of Auxilliary Groups
3. Framings of a subgroup $S \subsetneq PG$
4. Justified Pairs
5. Extension of framed subgroups to Justified, Hereditary Framed Groups
6. Use Zorn's lemma on the collection of all partial extensions of ϕ, and use 5. to show that $S = S'$ by setting $\hat{\phi}(s) = g$ where g is the game from 5. and $s \in S' \backslash S$.

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- Further, for sets of games A, and countable sets of countable sets of games $\{B_n\}$ and $\{C_n\}$, if for all $a \in A$, $a \not\leq b_1$ in B_1, then there exists a game x such that
  1. $a \not\leq x$ for all $a \in A$;
  2. $b_n \leq nx$ for all $b_n \in B_n$;
  3. $nx \not\leq c_n$ for all $c_n \in C_n$.

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  1. $S_i + S_j \subseteq S_{i+j}$
  2. $g \in S_0$ if and only if $g \geq 0$
- If $S \subseteq S' \trianglelefteq PG$, then any framing of S extends to S'.

Lurie's Proof (Justification)

- For a framed subgroup S, if $g \notin S_n$ for some $g \in PG$, then we say (g, n) is justified if there is an $x \in S_{-1}$ such that $g + x \notin S_{n-1}$.

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- From here we can find a framed subgroup S' extending S such that for *any* $g \in S_n$ but $g \in S$, for $n \neq -1, 0, 1$, the pair (g, n) will be justified in S'.

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- Consequently, for any framed subgroup S, there is a game x such that $S_n = \{y \in S \colon nx \leq y\}$ for every $n \in \mathbb{Z}$. This allows us to find extensions when we need to adjoin one element.

Lurie's Proof

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- Use framings and justified extensions to contradict maximality.

Thoughts on the h-hierarchy and the embedding result

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- The class of partizan games might be too large to be useful
- What subclasses might be useful, i.e. are there useful subgroups that form proper classes that can serve as useful homogeneous embedding objects.
- The surreal numbers are one candidate, but what would some others look like given the earlier structure results mentioned?