Vector Calculus and Stokes' Theorem Review

Alexander Berenbeim

May 6, 2018

The purpose of this review sheet is to gather and summarize much of the linear algebra and vector calculus that we've implicitly been working with this semester, and that I tried to detail throughout the discussion sections when time permitted, or when it was appropriate. Hopefully it will be useful in the time that you have left to study. That said, this review sheet often goes beyond the scope of the course, and so it is useful more as a road map of the theory behind the course material. Throughout this review sheet, I will use a, b, c to indicate real numbers. I will use f, g, h to indicate real valued functions, and F, G, H to indicate real-vector valued functions. I will use x, y, z as coordinate functions. Finally, all subscripts will refer to their letter assignment, i.e. a_1, a_2, b_1, b_2 , et cetera refer to real numbers, while f_1, f_2, f_3 will refer to functions.

1 Vectors and Vector Spaces

- In this course, we studied **Euclidean vector spaces**.
- We think of **vectors**, which are the elements of these spaces, as directions in some real-valued space of some **magnitude** (which we often denoted by $|\vec{v}|$ or $\|\vec{v}\|$ or $\|\mathbf{v}\|$), that we can add or substract (so if $\vec{v}$ and $\vec{w}$ are vectors, then $\vec{v}+\vec{w}$ is also a vector found by adding the component entries component wise), and scale by real numbers $c \in \mathbb{R}$ which in the general case are called **scalars** (this is what we mean by *real-valued*).
- All vector spaces have **bases**, which allow us to express vectors as sums. For example, in $\mathbb{R}^3$, we use the $a\vec{i}+b\vec{j}+c\vec{k}$ notation and $\langle a, b, c\rangle$ interchangeably.
- In general, we use orthonormal bases, which we can think of as the ordinary axes of $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, etc. In the literature, these bases are often expressed as $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and vectors $\vec{v} = \sum_{i=1}^{n} c_i\vec{e}_i$. In this course, we used $\vec{i}, \vec{j}$ and $\vec{k}$ for the basis vectors of $\mathbb{R}^3$.
- The advantage of using an explicit basis is that it makes summing and scaling vectors clearer. For example in $\mathbb{R}^3$, if $\vec{v} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{w} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ then

$$\vec{v}+\vec{w} = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k}) + (b_1\vec{i}+b_2\vec{j}+b_3\vec{k}) = (a_1+b_1)\vec{i}+(a_2+b_2)\vec{j}+(a_3+b_3)\vec{k}$$

  and for any $c \in \mathbb{R}$

$$c\vec{v} = ca_1\vec{i}+ca_2\vec{j}+ca_3\vec{k}$$

2 Linear Transformations

- A **linear transformation** is a map between two vector spaces which preserves vector addition and scalar multiplication. We denote this by $F : V \to W$, and the above means that $F(\vec{v}+\vec{w}) = F(\vec{v}) + F(\vec{w})$ and $F(c\vec{v}) = cF(\vec{v})$ for all vectors $\vec{v}, \vec{w}$ and real numbers c.
- We completely describe linear transformations by describing what they do to the bases of the two spaces. For example, we can define a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $\vec{e}_1 \mapsto \vec{e}_2$, $\vec{e}_2 \mapsto \vec{e}_3$ and $\vec{e}_3 \mapsto \vec{e}_1 + \vec{e}_2$. Then for any vector $\langle a, b, c\rangle$,

$$F(\langle a, b, c\rangle) = F(a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3) = F(a\vec{e}_1) + F(b\vec{e}_2) + F(c\vec{e}_3) = aF(\vec{e}_1) + bF(\vec{e}_2) + cF(\vec{e}_3)$$
$$= a\vec{e}2 + b\vec{e}_3 + c(\vec{e}_1 + \vec{e}_2) = \langle c, a + c, b\rangle$$

- We explicitly represent linear transformations with respect to a basis using a **matrix**. In the example above, F is represented by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so
$$A\vec{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+1\cdot 1 \\ 1\cdot 0+0\cdot 0+1\cdot 1 \\ 0\cdot 0+1\cdot 0+0\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1 + \vec{e}_2$$
as we defined above.

3 Vector-valued functions

- Throuhgout calculus I and II, we encountered ordinary real valued functions, usually of some form $f : [a, b] \to \mathbb{R}$, or $f : \mathbb{R} \to \mathbb{R}$.
- In this course, we first started looking at real valued functions with multiple variables, like $f : [0, 1] \times [0, 2\pi] \to \mathbb{R}$ where $f(r, \theta) = r\cos\theta$ or $f(r, \theta) = r\sin(\theta)$.
- We then started looking at vector-valued functions, i.e. functions that would send vectors to vectors.
- While linear transformations are vector valued functions, they're a very particular kind of vector valued function. Another example is the **cross-product** which sends vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$. Recall that the cross product was defined by by
$$\vec{v}\times\vec{w} = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3-a_3b_2)\vec{i}-(a_1b_3-a_3b_1)\vec{j}+(a_1b_2-a_2b_1)\vec{k}$$
- Another example of fundamental importance would be functions like $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, such that
$$\vec{r}(t) = \langle f_1(t), f_2(t), \cdots, f_n(t)\rangle$$
where each $f_i(t)$ is called a **component** functions. One example we often saw in this class $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$ defined on $[0, 2\pi]$. This function would start at $(1, 0)$ in $\mathbb{R}^2$, and then travel the unit circle counter-clockwise.
- In general, a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ will have the form
$$F(x_1, x_2, \ldots, x_n) = (F_1(x_1, x_2, \ldots, x_n), F_2(x_1, \ldots, x_n), \cdots, F_m(x_1, \ldots, x_n))$$
- These are all vector valued in the sense that they output a vector point wise.

4 Directional Derivatives

- One important example of vector spaces are **tangent spaces** to $\mathbb{R}^n$ at a point p, denoted by $T_p\mathbb{R}^n$.
- For example, in $\mathbb{R}^3$, the tangent space at a point $p \in \mathbb{R}^3$ is the space of vectors with the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$.
- Any vector $\vec{v} \in T_p\mathbb{R}^n$, called a **geometric tangent vector**, yields a map denoted by $D_{\vec{v}}|_p$ which sends differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ to $\mathbb{R}$ by taking the **directional derivative** in the direction $\vec{v}$ at the point p,
$$D_v|_p f = D_v f(p) = \frac{d}{dt}|_{t=0} f(a + tv) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_n}(p)$$
where $\vec{v} = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{x_2} + \cdots + a_n\frac{\partial}{\partial x_n}$.
- The definition agrees with the defintion of a directional derivative seen in class. If we let $\vec{v}$ denote an ordinary vector in $\mathbb{R}^n$, then the definition of the directional derivative at a point p in the direction of $\vec{v}$ is given by $\vec{v}\cdot\nabla f|_p = D_{\vec{v}}|_p(f)$, where ∇f denotes the gradient of f.

4.1 Vector Fields and Line Integrals

- Given a region $U \subset \mathbb{R}^n$, a map which sends each point in U to a tangent vector is called a **vector field on U**. That is, $X : U \to TU$ (where TU denotes the union of all the tangent spaces at points p of U) is given by $X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i}|_p$, where each $X_i : U \to \mathbb{R}$ is a **component function of X**.
- Vector fields are one example of vector valued functions, and in fact, they're ones we've studied quite a bit in this course.
- One way to get a vector field is take the gradient of a function (in fact, all such vector fields are called **conservative**), i.e. if there is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = X$, then
- If $U \subset \mathbb{R}^n$ and $\gamma : [a, b] \to U$ is a curve inside U (i.e. γ is vector valued function whose image lies inside U given by $\gamma(t) = \langle x_1(t), \dots x_n(t)\rangle)$, then we define the **line integral of X over** γ for all differentiable vector fields X on U by

$$\int_\gamma X \cdot ds = \int_a^b X_{\gamma(t)} \cdot \gamma'(t)dt$$

where

$$\gamma'(t) = \begin{bmatrix} \frac{dx_1}{dt} \cdots \frac{dx_n}{dt} \end{bmatrix}$$

- For example, for $S \subset \mathbb{R}^2$ is the unit square, $\gamma : [0, 1] \to U$ is given by $\gamma(t) = \langle t^2, t\rangle$, and we consider the vector field given by

$$X = (x + y)\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \langle x + y, -1\rangle,$$

then

$$\int_\gamma X \cdot ds = \int_0^1 X_{\gamma(t)} \cdot \gamma'(t)dt = \int_0^1 \langle t^2 + t, -1\rangle \cdot \langle 2t, 1\rangle dt = \int_0^1 2t^3 + 2t^2 - 1dt = \frac{1}{2} + \frac{2}{3} - 1 = \frac{1}{6}$$

since $X_{\gamma(t)} = \langle x(t) + y(t), -1\rangle = \langle t^2 + t, 1\rangle$.

5 The differential and the Jacobian

- If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then for all $p \in \mathbb{R}^n$, the **differential of F at p** is the linear transformation map

$$dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$$

defined by the rule

$$dF_p(\vec{v})(f) = \vec{v}(f \circ F)$$

where $\vec{v} \in T_pM$ is a tangent vector.

- With the basis of $T_p\mathbb{R}^n$ given by

$$\{\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p, \cdots, \frac{\partial}{\partial x_n}|_p\}$$

and the basis of $T_{F(p)}\mathbb{R}^m$ given by

$$\{\frac{\partial}{\partial y_1}|_p, \frac{\partial}{\partial y_2}|_p, \cdots, \frac{\partial}{\partial y_m}|_p\}$$

, then the action of dF_p is defined on a typical basis vector $\frac{\partial}{\partial x_i}$ and differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ as

$$dF_p(\frac{\partial}{\partial x_i}|_p) = \frac{\partial}{\partial x_i}|_p(f \circ F) = \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(F(p))\frac{\partial F_j}{\partial x_i}(p) = \left(\sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial y_j}\right) f$$

so then

$$dF_p = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_1}{\partial x_n}(p) \\ & \ddots & \\ \frac{\partial F_m}{\partial x_1}(p) & \cdots & \frac{\partial F_m}{\partial x_n}(p) \end{pmatrix}$$

- In fact, when $n = m$, or more generally, when $F : U \to V$ where $U, V \subset \mathbb{R}^n$, then dF_p is just the Jacobian matrix as seen in class.
- For example, recall the first exercise from the week 12 discussion: We want to evaluate the integral of $f(x, y) = x^2 y$ over the region $R = \{(x, y) \mid 0 \leq x \leq 2, x \leq y \leq x + 4\}$. We are told to consider the transformation $F : S \to R$ where $x = 2u$ and $y = 4v + 2u$. Then $F = \langle 2u, 2u + 4v\rangle = \langle f_1(u, v), f_2(u, v)\rangle$ with $S = \{(u, v) \mid 0 \leq u, v \leq 1\}$, and thus

$$J_F(u, v) = dF = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}$$

  We will return to this example after we discuss co-vectors and line integrals.

- It should be noted that in class, the determinant of the Jacobian as the Jacobian.

6 Co-vectors

- For a vector space V, a **linear functional** is a map $F : V \to \mathbb{R}$ such that $F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$ and $F(c\vec{v}) = cF(\vec{v})$. You've seen them in this class whenever we take a dot product between two vectors $\vec{v}$ and $\vec{w}$. We can let $F_{\vec{v}}(\vec{w}) = \vec{v} \cdot \vec{w}$.
- A **covector** on a vector space is a real-valued linear functional. The space of linear functionals on a vector space V is often denoted V^* and is called the **dual space of V**. Covectors also form a vector space, and if V is finite dimensional, then V^* is of the same dimension.
- One very important class of dual spaces are the dual spaces to tangent spaces, called **cotangent spaces**. The **cotangent space of U at p** is denoted by T_p^*U.
- Elements of cotangent spaces are called covectors or cotangent vectors, and are typically denoted by ω. Since T_p^*U is a vector space, it has a basis. By preference, we express T_p^*U with a basis **dual** to T_pU, suggestively given by

$$\{dx_1, \ldots dx_n\},$$

  so that

$$\frac{\partial}{\partial x_i} dx_j = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- With the dual basis given above, a covector can be written as a sum $\omega = a_1 dx_1 + a_2 dx_2 + \cdots a_n dx_n$.
- Just as we defined vector fields, we can describe **covector fields** or **differential 1-forms** as differentiable maps from a space U to its space of cotangent vectors, where

$$\omega = g_1 dx_1 + \cdots + g_n dx_n$$

  with component functions $g_1, \ldots, g_n$.

- Just as we used the gradient of a function f to define a vector field, we can define a covector field denoted by df (and called the **differential** of f) which is defined by

$$df_p(v) = vf$$

for all $v \in T_pM$. Concretely,

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)dx_i|_p$$

- The derivative of a function f along a curve γ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t))$$

- If ω is a differential 1-form and X a vector field on U, then we can form a function $\omega(X) : U \to R$ given by $

$$\omega(X)(p) = \omega_p(X_p) = \sum_{i=1}^{n} g_i(p) \cdot f_i(p)$$

where the f_i are the component functions of a vector field.

6.1 Line Integrals of Covectors and the Fundamental Theorem of Line Integrals

- Given a covector ω on V, and a differentiable function $F : U \to V$, we define the **pullback of ω by F** pointwise by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}) = \omega_{F(p)}(dF_p(\vec{v}))$$

where $\vec{v} \in T_pU$. So if $F : S \to R$ is a differentiable function, and $f : R \to \mathbb{R}$ is a continuous function, and ω is a covector field on R, then $F^*(f\omega) = (f \circ F)F^*\omega$.

- While you likely won't see much use in the above expression, it will come back in a bit.

- Given a differentiable covector field on U ω, and a curve $\gamma : [a, b] \to U$, we define the **line integral of ω over** γ to be the real number found by

$$\int_\gamma \omega = \int_a^b \gamma^*\omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt$$

- (**Fundamental Theorem of Line Integrals**) If $\omega = df$ and γ is a curve defined on domain $[a, b]$, then

$$\int_\gamma \omega = \int_\gamma df = \int_a^b \gamma^*df = \int_a^b df_{\gamma(t)}(\gamma'(t))dt = \int_a^b (f \circ \gamma)'(t)dt = f(\gamma(b)) - f(\gamma(a))$$

7 n-forms

- An important consequence of covectors being maps of vectors to $\mathbb{R}$ is that we can extend them to **multi-covectors** or **differential forms** by means of an **alternating** (or **antisymmetric** or **skew-symmetric**) multiplication called the **exterior** or **wedge** product, denoted by $\wedge$. By alternating we mean that given two covectors dx and dy, $dx \wedge dy = -dy \wedge dx$. As a consequence $dx \wedge dx = 0$.

- We let $\Omega(\mathbb{R}^m)$ be the space of all differential forms, and we let $\Omega^n(\mathbb{R}^m)$ be the subspace of all **n-forms**, which are sum-products of functions with wedges of n-distinct coordinates.

- For example:
  - a 0-form is an ordinary differentiable function $f : \mathbb{R}^m \to \mathbb{R}$.
  - A 1-form is a sum of the form $f_1dx_1 + f_2dx_2 + \cdots + f_mdx_m$.
  - A 2-form is a sum of the form $\sum_{1 \le i < j \le m} f_{i,j}dx_i \wedge dx_j$.

- Crucially, $\Omega^n(\mathbb{R}^m) = 0$ for all $n > m$. When $n = m$, the members of $\Omega^m(\mathbb{R}^m)$ are called **top-forms**. They are very important, because they correspond with our notions of integrating functions with respect to area, volume, etc.
- An important class of 2-forms are the surface forms. In $\Omega^2(\mathbb{R}^2)$, these are precisely the forms $f(x,y)dx \wedge dy = f(x,y)dA$, which are called **area forms**
- If $\omega \in \Omega^2(\mathbb{R}^3)$, then
$$\omega = f_1 dydz + f_2 dzdx + f_3 dxdy$$
- If $\omega \in \Omega^3(\mathbb{R}^3)$, then $\omega = fdxdydz = fdV$, and is called a **volume form**.

8 The Differential and the Jacobian (revisited)

- The differential d seen above is a map $\Omega^0(\mathbb{R}^m) \to \Omega^1(\mathbb{R}^m)$. We can extend d as a map to all $\Omega^{n-1}(\mathbb{R}^m) \to \Omega^n(\mathbb{R}^m)$ as follows: given $\omega \in \Omega^{n-1}(\mathbb{R}^m)$,

$$\begin{aligned} d\omega &= d\Big(\sum_{1\le i_1<i_2<\cdots<i_{n-1}\le m} f_{i_1,\ldots,i_{n-1}} dx_{i_1} \wedge \cdots dx_{i_{n-1}}\Big) \\ &= \sum_{1\le i_1<i_2<\cdots<i_{n-1}\le m} d(f_{i_1,\ldots,i_{n-1}}) dx_{i_1} \wedge \cdots dx_{i_{n-1}} \\ &= \sum_{1\le i_1<i_2<\cdots<i_{n-1}\le m} \left(\sum_{j=1}^{m} \frac{\partial f_{i_1,\ldots,i_{n-1}}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_{n-1}} \right) \end{aligned}$$

and whenever dx_j appears in $dx_{i_1} \wedge \cdots dx_{i_{n-1}}$, then $dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_{n-1}} = 0$. Otherwise, we can permute the indices to be in ascending order and as a result, we can flip the signs of our entry.

- Let's work through some clear examples that we see in this course:
- (**Gradient**) If $f \in \Omega^0(\mathbb{R}^n)$, then
$$df = \nabla f \cdot \langle dx_1, \ldots, dx_n \rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$
- If $\omega \in \Omega^1(\mathbb{R}^2)$, then $\omega = P(x,y)dx + Q(x,y)dy$, and then

$$\begin{aligned} d\omega &= d(P(x,y))dx + d(Q(x,y))dy \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}\right) dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) dy \\ &= \frac{\partial P}{\partial x}dx \wedge dx + \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial y}dy \wedge dy \\ &= \frac{\partial Q}{\partial x}dx \wedge dy - \frac{\partial P}{\partial y}dx \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \end{aligned}$$

- (**Curl**) If $\omega \in \Omega^1(\mathbb{R}^3)$, then $\omega = Pdx + Qdy + Rdz$ and

$$
\begin{aligned}
d\omega &= d(Pdx + Qdy + Rdz) \\
&= (dP)dx + (dQ)dy + (dR)dz \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) dz \\
&= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial z}dz \wedge dy + \frac{\partial R}{\partial x}dx \wedge dz + \frac{\partial R}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \\
&= F \cdot d\vec{S}
\end{aligned}
$$

- (**Divergence**) If $\omega \in \Omega^2(\mathbb{R}^3)$, then $\omega = Pdydz + Qdzdx + Rdxdy$ and

$$
\begin{aligned}
d\omega &= (dP)dydz + (dQ)dzdx + (dR)dxdy \\
&= \frac{\partial P}{\partial x}dxdyz + \frac{\partial Q}{\partial y}dydxdz + \frac{\partial R}{\partial z}dzdxdy \\
&= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdyz \\
&= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV
\end{aligned}
$$

8.0.1 The Jacobian Revisited

- While the pullback formula for differential forms might seem mysterious, the only one that matters in this course will be **pullbacks for top-degree forms**.
- Explicitly, when $F : U \to V$ is a differentiable map where $(x_1, \dots x_n)$ are the coordinates of U and $(y_1, \dots y_n)$ are the coordinates of V, and $f : V \to \mathbb{R}$ i sa differentiable map,

$$F^*(fdy_1 \wedge dy_2 \wedge \cdots d_{y_n}) = (f \circ F)(\det dF)dx_1 \wedge \cdots \wedge dx_n$$

- So for our example from before where $F : S \to R$ given by $F(u, v) = \langle 2u, 2u + 4v\rangle$, and$f(x, y) = x^2y$, the pullback formula on the top degree form is given by

$$F^*(fdx \wedge dy) = (f \circ F)(\det dF)du \wedge dv$$

or explicitly by

$$(f \circ F)8du \wedge dv = 8(4u^2)(2u + 4v)dudv = 16(4u^3 + 8u^2v)dudv$$

which we would then integrate on the unit square S.

- This also works for changing from spherical to cartesian coordinates, or from polar to cartersian (give it a try for practice).

9 Manifolds and Stokes' Theorem

- While this is not the complete definition of a **manifold**, for the purposes of this course, a **manifold** M is a space such that for all $p \in M$, there is a subset U of M containing p that 'looks' like a subset of $\mathbb{R}^n$ for some fixed n. We have seen many examples of manifolds, most notably, every $\mathbb{R}^n$ is a manifold.
- $\mathbb{N} = \{0, 1, 2, \dots\}$ is a 0-dimensional manifold.
- $(0, 1)$ and $[0, 1]$ are both 1-dimensional manifolds. So is $S^1 = \{(\cos t, \sin t) \mid t \in R\}$, since even though $S^1 \subset \mathbb{R}^2$, every point $p \in S^1$ is contained in a subset of S^1 that can be bijective mapped to an interval in $\mathbb{R}^1$.
- $[a, b] \times [c, d]$ is a 2-manifold.

- $[0,1]\times[0,1]\times[0,1]$ is a 3-manifold, and so on.
- In the case of $M=[0,1]$, we say that the **boundary** of M, denoted by ∂M is the 0-manifold $\{0,1\}$. In the case of $M=[a,b]\times[c,d]$, where $a<b$ and $c<d$, the boundary ∂M is the union of subsets

$$\partial M=[a,b]\times\{c\}\cup\{b\}\times[c,d]\cup[b,a]\times\{d\}\cup\{a\}\times[d,c]$$

  where $[b,a]=[a,b]$ as a set, but suggests the orientation of traveling from b to a instead of from a to b (similarly for $[d,c]$).
- In this course, we parametrize our boundaries with curves $\gamma(t)$ in this case of boundaries of surfaces, or with some 2-form in this case of 3 dimensional manifolds.
- In general, if M is an n-manifold with boundary, then ∂M is an (n-1)-manifold with boundary. This is because in manifold with boundary, all points are either locally in sets that can be contained in $\mathbb{R}^n$ or in $\mathbb{H}^n=\{(x_1,\dots,x_n)\mid x_n\geq 0, x_1,x_2,\dots,x_{n-1}\in\mathbb{R}\}$, where if $p\in M$ is a boundary point, then it is bijectively in correspondence with a point $(x_1,\dots,x_{n-1},0)$, which is in bijective correspondence with $\mathbb{R}^{n-1}$.
- In this course, all the spaces we look at are oriented manifolds, and so I will omit the definition of orientation to avoid further frustration and confusion.
- The general form of **Stokes theorem** states that for an oriented n manifold M with boundary, if $\omega\in\Omega^{n-1}(M)$, then

$$\int_{\partial M}\omega=\int_M d\omega$$

9.1 Green's Theorem

For Green's theorem, M is a surface in 2-dimensions whose boundary is parametrized by some curve γ, and $\omega=Pdx+Qdy$, so that

$$\int_\gamma Pdx+Qdy=\int_M\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

9.2 Stokes' Theorem (Surface Integral)

If S is a surface with boundary and $F=P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}$ is a vector field on $\mathbb{R}^3$, and $r(t)$ describes the boundary $\partial S=C$, then

$$\int_C F\cdot dr=\int_{\partial S}Pdx+Qdy+Rdz=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dy\wedge dz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dz\wedge dx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy=\iint_S F\cdot dS$$

9.3 Divergence Theorem

If V is some three dimensional manifold whose boundary is the surface S, and F is a differentiable vector field on V, then $\iiint_V(\nabla\cdot F)dV=\iint_S(F\cdot\hat{n})dS$. Alternatively, we can note that $F\cdot\hat{n}dS$ defines a 2-form ω where

$$\omega=F_1dydz+F_2dzdy+F_3dxdy$$

so that

$$d\omega=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$$

10 One last aside...

Say you are given some surface $z=f(x,y)$ say $z=9-x^2-y^2$ where $z\geq 0$, so that the boundary is given by $x^2+y^2=9$, then for a vector field X such that $curlX=F$, since $dz=d(9-x^2-y^2)=-2xdx-2ydy$

$$X\cdot dS=d\omega=F_1dydz+F_2dzdy+F_3dxdy=F_1dy(-2xdx-2ydy)+F_2(-2xdx-2ydy)dx+F_3dxdy=(2xF_1+2yF_2+F_3)dxdy$$

which can be evaluated by transforming to polar coordinates.