A GENTLE INTRODUCTION TO TOPOS THEORY AND COHESIVE HOMOTOPY TYPE THEORY

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Abstract. The generalization of *cohesive topos theory* is among homotopy type theory's many promising aspects. *Cohesion* can be understood as generalizing a notion of geometric structure, and the implementation of homotopy type theory in languages such as `Agda` and `Coq` enables automated proof checking for formerly inaccessible objects such as line bundles. Meant as a mostly self contained introduction to the subject this paper is divided into 2 somewhat asymmetrical sections. Section I provides a more or less complete account of the categorical tools and vocabulary necessary to understand cohesion with a type theoretic flavour. Section II elaborates on what is meant by cohesion, focusing on Lawvere's seminal paper [11] on axiomatic cohesion, and provides two rather illuminating examples which are developed throughout Section I. Intended as a self-contained exposition and introduction to the subject, this paper aims to equip readers with the vocabulary and some working intuition when approaching current open problems in cohesive homotopy type theory. No new results are provided, and several of the proofs presented contain gaps meant for the reader to fill.

Contents

1. Preliminaries

This section exists for readers who are unfamiliar with category theory and topos theory. If you are already familiar with this material, you will be able to skip ahead to section II without difficulty. The definitions are incrementally provided in order to develop cohesion from its essential, irreducible pieces.[1] Along the way, an immense amount of categorical machinery will be introduced, but always with the motivation of providing sense to the following idea:

> Cohesion is an adjoint triple of modalites over a category of objects possessing a structure of interest, which also has a rich enough internal structure such that one can do mathematics, and where the modalities identify distinct full subcategories such that the composition of modalities expresses a dynamic transformation of the structure of interest.

What exactly is meant by modalities, internal structure of interest (and one that allows one to do mathematics), as well as full subcategories will all be explored, and given several concrete examples. Additionally, a multitude of footnotes are provided for those readers who may not be familiar with homotopy type theory or other categorical ideas whose introduction would interrupt the *flow* of the paper.

The central example developed throughout the paper is the case of *directed reflexive graphs*. The exposition of this category, from its introduction as a category in its own right, to a subcategory of a specific presheaf category, to a topos, to a coherent topos, only incidentally happened to lend this paper a spine similar to the core concept of *cohesion*.

In all, this section should be read as a rather self-contained collection of definitions, remarks and examples such that a reader who has some passing familiarity with type theory, category theory, or homotopy theory can find an "in" to the wonderful world of cohesive homotopy type theory.

Furthermore, although many of these definitions, and a fuller exposition are given in [1, p. 383] with a type theoretic flavour, readers who are still lost are recommended to look at [5], [9] and [6]. [5] is a great text if you are coming to category theory as a mathematician; [9] is great if you're coming to category theory with an interest in logic; [6] is a friendly introduction if you prefer a computer science perspective to this material.[2] Moreover, the definitions presented here describe how category theory can be realized within type theory.

1.1. Categories, Functors, and Natural Transformations In HoTT .

Definition. A **precategory** A consists of :

1. A type A_0 of **objects**. If a is an object of A_0, we may denote this by $a : A$;
2. for each $a, b : A$, a set[3] $\mathtt{Hom}_A(a, b)$ of **morphisms** $f : a \to b$;
3. for each $a : A$, a morphism $1_a : \mathtt{Hom}_A(a, a)$;
4. for each $a, b, c : A$, a function $g \circ f : \mathtt{Hom}_A(b, c) \to \mathtt{Hom}_A(a, b) \to \mathtt{Hom}_A(a, c)$;
5. for each $a, b : A$ and $f : \mathtt{Hom}_A(a, b)$, $f = 1_b \circ f$ and $f = f \circ 1_a$;
6. for each $a, b, c, d : A$ and $f : \mathtt{Hom}_A(a, b), g : \mathtt{Hom}_A(b, c)$, and $h : \mathtt{Hom}_A(c, d)$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition. $f : \mathtt{Hom}_A(a, b)$ is an **isomorphism** if there is a $g : \mathtt{Hom}_A(b, a)$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$. The type of these isomorphisms is denoted by $a \cong b$.

Since all hom-sets are sets, the identity types of hom-sets are mere propositions. With a little effort[4], it can be shown that $f : a \cong b$ is a mere proposition. Moreover, if A is a precategory and $a, b : A$, then by induction on identity, we have $\mathtt{idtoiso} : (a = b) \to (a \cong b)$, which recognizes that a path between objects is an isomorphism.

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[1] In this case, distinct categorical constructions.

[2] Though the perspective is mathematical, many of Awodey's exercises will feel naturally motivated for a computer scientist.

[3] In category theory, this can very well be a class of morphisms $f : a \to b$, although for the purposes of cohesion, having the class of morphisms between objects behave as sets suffices.

[4] Supposing we have $f : \mathtt{Hom}_A(a, b), g, g' : \mathtt{Hom}_A(b, a), \eta, \eta' : (1_a = g \circ f), \varepsilon, \varepsilon' : (f \circ g = 1_b)$,

$$g = g \circ 1_b = g \circ (f \circ g') = (g \circ f) \circ g' = 1_a \circ g' = g'$$

using ε', η. This establishes that $g = g'$ and moreover that $(g, \eta, \varepsilon) = (g', \eta', \varepsilon')$. Moreover, the type $a \cong b$ is a set.

Definition. A **category** is a precategory where `idtoiso` is an equivalence.[5] So, by the univalent axiom, we have that `Set` is a category, as are any precategories with set-level structures.

Here are some interesting categories which will appear throughout this paper:

Example. The category of groups, denoted `Grp`, has groups as objects, and group homomorphisms as morphisms. This shouldn't surprise the reader, as $\mathtt{Hom}_{\mathtt{Grp}}(a, b)$ is the hom-set we all know and love.

Although it can be described as a subcategory of `Grp`, due to its ubiquity, we often find the need to work with the category of abelian groups, which is denoted by `Ab`. Clearly the objects are abelian groups, and the morphisms are group homomorphisms.

From the category of abelian groups, it is natural to find the category of rings,[6] denoted by `Ring`. This category has rings as its objects and ring homomorphisms as its morphisms. These rings need not be commutative. For that, one can work within the category of commutative rings `CRing`.

Example. One very important category is the category of topologies, denoted by `Top`. Objects $(a, \tau) : \mathtt{Top}$ are pairs consisting of $a : \mathtt{Set}$ and a topology τ on a, satisfying the axioms of topology. The morphisms of `Top` are continuous maps.

For geometers, the following related categories are incredibly important:

Example. The category of smooth manifolds, denoted by `Diff`, has paracompact smooth manifolds as objects and smooth functions for morphisms. A closely related category is the category of cartesian spaces, denoted by `CartSp`, whose objects are the cartesian spaces $\mathbb{R}^n$, for $n \in \mathbb{N}$, and whose morphisms consist of suitable, structure preserving functions between these spaces.

It is common in the literature of this subject to see `CartSp` with a subscript indicating the structure of cartesian spaces which these functions preserve. For instance $\mathtt{CartSp}_{smooth}$ regards $\mathbb{R}^n$ as smooth manifolds with smooth functions as morphisms, while $\mathtt{CartSp}_{lin}$ regards $\mathbb{R}^n$ as real vector spaces, and has linear functions as morphisms.

Example. Supposing that X is a topological space with topology τ, we can turn X into a category $\Theta(X)$ by taking the open sets of X as objects and the inclusion relation $\subseteq$ as the only morphism.

The enthusiastic reader can indeed verify that this forms a category, as indeed any partially ordered set forms a category.

Example. An example that we will return to throughout this paper is the category of reflexive graphs, denoted by `RGph`.

Now given that graphs have a bit of ambiguity in how they're described, in the most general permissible sense, the objects are graphs G, which consist of triples (V, E, d). V is a set of *vertices*, E is a set of *edges*, and $d : E \hookrightarrow V \times V$. Moreover, for every $a : V$, there is an edge $e_a : E$ such that $d(e_a) = (a, a)$ (from which we get the reflexivity).

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[5] For readers unfamiliar with the type theoretic notion of equivalence, this is not the same as an equivalence relation! Recall that given a map $f : A \to B$, **quasi-inverse** of f is a triple (g, α, β) consisting of a map $g : B \to A$ and homotopies $\alpha : f \circ g \sim \mathtt{id}_B$ and $\beta : g \circ f \sim \mathtt{id}_A$. Moreover, the type of quasi-inverses of f is

$$\mathtt{qinv(f)} :\equiv \sum_{g:B\to A} ((f \circ g \sim \mathtt{id}_B) \times (g \circ f \sim \mathtt{id}_A))$$

while equivalence from $A \to B$ is a pair consisting of $f : A \to B$ and $(A \simeq B)$, where (by abuse of notation)

$$(A \simeq B) : \mathtt{isequiv(f)} :\equiv \left(\sum_{g:B\to A} (f \circ g \sim \mathtt{id}_B)\right) \times \left(\sum_{h:B\to A} (h \circ f \sim \mathtt{id}_A)\right),$$

which has the following properties:

(i) for each $f : A \to B$, there is a function $\mathtt{qinv}(f) \to \mathtt{isequiv}(f)$;
(ii) for each $f : A \to B$, there is a function $\mathtt{isequiv}(f) \to \mathtt{qinv}(f)$;
(iii) For any two $e_1, e_2 : \mathtt{isequiv}(f)$, there is a path $e_1 = e_2$.

It should be noted that $\left(\sum_{g:B\to A} (f \circ g \sim \mathtt{id}_B)\right) \times \left(\sum_{h:B\to A} (h \circ f \sim \mathtt{id}_A)\right)$ is only one of the easiest such definitions which satisfies those properties; equivalences are defined by those properties. For a fuller account of equivalences from a type theoretic perspective, refer to [1, p 162-175]

[6] One should be able to recall that rings are abelian groups with additional structure.

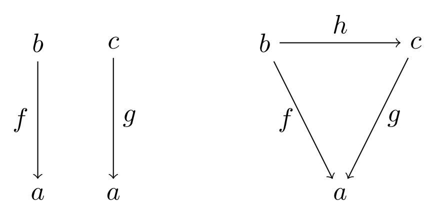
When reasoning about different graphs, we denote the vertices of a graph G by $G(V)$ and the edges of a graph G by $G(E)$.

Furthermore, we may omit the d map, and consider $G(E)$ as a set such that given a source and target mapping $s, t : G(E) \to G(V)$, we can identify e with a pair $\{a, b\}$ where $a, b : G(V)$ and $a = s(e)$ and $b = t(e)$.[7] In what might seem like silly solipsism, but keeping in spirt with the idea that E is an embedding in the cartesian product of $G(V) \times G(V)$, we note $e = \{s(e), t(e)\}$.

Morphisms of $\texttt{RGph}$ are graph homomorphisms $f : G \to H$, i.e. a pair of morphisms $(f_1 : G(V) \to H(V), f_2 : G(E) \to H(E))$ such that $f_1(G(V)) \subseteq V(H)$ and if $\{a, b\} : G(E)$ then $\{f_1(a), f_1(b)\} : H$ (that is to say, $f_2(\{a, b\}) = \{f_1(a), f_1(b)\}$).

Although it might be helpful to keep this construction in mind, we will revisit this category again, from a purely categorical point of view.[8]

Example. Given any category A and an object $a : A$, the **slice category** $A \downarrow a$ has the morphisms with a as the target as its objects, i.e. $f : b \to a$ are the objects of a slice category, and for objects $f : b \to a$ and $g : c \to a$, the morphisms $h : b \to c$ such that $f = h \circ g$ are the morphisms of the slice category. It is worth illustrating why categories constructed in this matter earn this name; let $a, b, c : A$ and let $f, g : A \downarrow a$, such that $f : b \to a$ and $g : a \to c$, then h is a morphism of $A \downarrow a$ if



commutes.[9]

Dually, one can define a **co-slice category** by $a \downarrow A$ where the objects are the morphisms of A whose source is a. For an instance of the utility of these constructions, consider for any commutative ring object $R : \texttt{CRing}$ the subcategory of the co-slice category $R \downarrow \texttt{Ring}$ where the objects are ring homomorphisms $f : R \to A$, such that $A : \texttt{Ring}$ with unity, $f \circ 1_R = 1_A$, and $f(R) \subseteq Z(A)$, where $Z(A)$ is the centre of A. This defines the category of R-algebras!

Example. Given a category A, a **full subcategory** S of A is given by

(1) A type S_0 of object, often denoted by S such that if $a : S$ then $a : A$;
(2) for each $a, b : S$, a set $\texttt{Hom}_S(a, b)$ of morphisms;
(3) for each $a : S$, a morphism $1_a : \texttt{Hom}_S(a, a)$;
(4) for each $a, b, c : S$, a function $g \circ f : \texttt{Hom}_S(b, c) \to \texttt{Hom}_S(a, b) \to \texttt{Hom}_S(a, c)$;
(5) for each $a, b : S$ and $f : \texttt{Hom}_S(a, b)$, $f = 1_b \circ f$ and $f = f \circ 1_a$;
(6) for each $a, b, c, d : S$ and $f : \texttt{Hom}_S(a, b), g : \texttt{Hom}_S(b, c)$ and $h : \texttt{Hom}_S(c, d)$, $h \circ (g \circ f) = (h \circ g) \circ f$;
(7) for each $a, b : S$, $\texttt{Hom}_S(a, b) = \texttt{Hom}_A(a, b)$.

Conditions (1)-(6) determine a subcategory, while condition (7) in particular determines that S is a full subcategory of A.

Moreover, there is a clear mapping from S to A, called the **inclusion functor**, which is an obvious faithful functor taking objects and morphisms to themselves.

Definition. Let A, B be precategories. A **covariant functor** $F : A \to B$ consists of:

(1) A function $F_0 : A_0 \to B_0$, denoted by F;
(2) for each $a, b : A$, a function $F_{a,b} : \texttt{Hom}_A(a, b) \to \texttt{Hom}_B(F(a), F(b))$, also generally denoted by F;
(3) for each $a : A$, $F(1_a) = 1_{F(a)}$;
(4) for each $a, b, c : A, f : \texttt{Hom}_A(a, b), g : \texttt{Hom}_A(b, c), F(g \circ f) = F(g) \circ F(f)$.

We similarly define a **contravariant functor** F by replacing (2) and (4) in the above definition as follows:

[7]Depending on whether we wish these graphs to be directed or undirected, we can insist on these pairs being ordered or not
[8]If you are a graph theory purist who finds this notation incorrect, please be patient. All will be motivated in time.
[9]Given the shape of the commuting diagram that the morphisms of this category satisfy, one can speculate about what foodstuff inspired this category's name.

(2') for each $a, b : A$, a function $F_{a,b} : \texttt{Hom}_A(a, b) \to \texttt{Hom}_B(F(b), F(a))$, also generally denoted by F;
(4') for each $a, b, c : A, f : \texttt{Hom}_A(a, b), g : \texttt{Hom}_A(b, c), F(g \circ f) = F(f) \circ F(g)$.

Example. (**Forgetful Functors**) Perhaps the most intuitive examples of functors are those which take the objects of category A to their underlying sets. We denote this functor by U. So for instance, both $U : \texttt{Top} \to \texttt{Set}$ and $U : \texttt{Grp} \to \texttt{Set}$ are forgetful functors which take topological and group objects respectively to their underlying sets, and "forgetting" the structure-preserving property of their morphisms, while effectively embedding their hom-sets.

More sophisticated forgetful functors, such as $U : \texttt{CRing} \to \texttt{Ab}$ which take commutative rings to their underlying abelian groups and in the process "forgetting" the multiplication preserving properties of ring homomorphisms, while preserving ring addition properties.

Example. Functors need not always lose information!

In some cases, functors can be thought of as adding structure. For instance, let $F, G : \texttt{Set} \to \texttt{Top}$ be two functors taking objects $a : \texttt{Set}$ to the discrete and indiscrete topology of $(a, \tau) : \texttt{Top}$ respectively.

To see how F, G act on $\texttt{Hom}_A(a, b)$, let $a, b : \texttt{Set}$ and $f : \texttt{Hom}_A(a, b)$. In both cases, any set function f is made into a continuous function. For instance, in the indiscrete case, it is simply[10] a matter of noticing that $[G(f)]^{-1}(b) \subseteq a$, whence $[G(f)]$ is continuous, as was to be expected.[11]

This is also very easy to check in the discrete case, since the inverse image of any subset V can be identified with the points mapping into V under f. Furthermore, the functoriality of F, G is easy enough to check, and so it will be left to the interested reader.

The reader is also encouraged to verify that $\texttt{Hom}_{\texttt{Set}}(a, b) \cong \texttt{Hom}_{\texttt{Top}}(F(a), F(b)) \cong \texttt{Hom}_{\texttt{Top}}(G(a), G(b))$. That is, F, G are full and faithful functors.

Example. One very important example of a contravariant functor is a **presheaf** F on a category A, given as $F : A \to \texttt{Set}$. It often makes more sense to write $F : A^{op} \to \texttt{Set}$ to convey the contravariance[12] of this functor (and to avoid confusion with the typical forgetful functors).

For instance, let X be a topological space. A set-valued presheaf of $\Theta(X)$ is the functor $F : \Theta(X)^{op} \to \texttt{Set}$ which takes the subsets of X to themselves, and maps $A \subseteq B$ to $f : B \to A$. In particular, where X is a terminal[13] object in $\Theta(X)$, in the subcategory defined by this presheaf, $F(X) = X$ is an initial object, and the map $F(\subseteq) : \texttt{Set}$ can be identified with the restriction map $\rho : \texttt{Set}$.

Definition. By induction on identity, we find that a functor preserve `idtoiso`. Now, if $F, G : A \to B$ are functors, then a **natural transformation** $\alpha : F \Rightarrow G$ consists of

(1) (components) for each $a : A$, $\alpha_a : \texttt{Hom}_B(F(a), G(a))$;
(2) (naturality) for each $a, b : A$ and $f : \texttt{Hom}_A(a, b)$, $G(f) \circ \alpha_a = \alpha_b \circ F(f)$.

Although covering bi-categories is well beyond the scope of this paper, it is somewhat instructive to note that one can compose functors and natural transformations.

Definition. Following [1], let A, B, C be categories and let $F : A \to B$ and $G, H : B \to C$ be functors with $\alpha : G \to H$. Then the **composite** $(\alpha \diamond_r F) : G \circ F \to H \circ F$ is a natural transformation with component $\eta_{F(a)}$ for each $a : A$. Similarly if $G, H : B \to A$ instead, then the composite $(F \diamond_l \alpha) : (F \circ G) \to (F \circ H)$ has components $F(\alpha_b)$ for each $b : B$.

Example. One intuitive example of natural transformations comes from studying the following **endofunctors** $1_{\texttt{Set}}$, the identity endofunctor, and the endofunctor $- \times 1 : \texttt{Set} \to \texttt{Set}$, defined as follows:

$$a : \texttt{Set} \mapsto a \times 1$$

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[10] If one is truly pedantic, one can also check the preimage of the empty set $\emptyset$.

[11] It should be noted that this is generally not the case with indiscrete topologies, as any time we have a map from $f : a \to b$, if b is a space which satisfies the T_0 axiom, then the only continuous functions are constant maps. However, since indiscrete topologies very clearly don't satisfy this except in the most trivial cases, we're golden!

[12] Indeed, to a category theorist, one really doesn't need to say contravariant at all, and just note that the definition of contravariance given above really only describes the dual category of A, denoted A^{op}, where objects of A^{op} are the objects of A, and the morphisms of A^{op} are the morphisms of A with the source and target switched, i.e. $f : a \to b$ in A^{op} is the same as $f : b \to a$ in A.

[13] i.e., for all objects $a : \Theta(X)$, there is a unique map $a \to X$, namely $a \subseteq X$ by definition. Similarly, we say that $\emptyset$ is initial in $\Theta(X)$, since for all $a : \Theta(X)$, there is a unique map from $\emptyset$ to a, namely $\emptyset \subseteq a$. For a full discussion on initial and terminal objects, consult [6] or [9].

and

$$f : a \to b \mapsto f \times 1 : a \times 1 \to b \times 1$$

which takes sets a to the product $a \times 1$ where 1 is a terminal object in set (namely, any arbitrary singleton). To see the natural transformation $\alpha : 1_{\texttt{Set}} \to - \times 1$, consider:

$$\begin{array}{ccccc} a & & a & \xrightarrow{\alpha_a} & a \times 1 \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ b & & b & \xrightarrow[\alpha_b]{} & b \times 1 \end{array}$$

Now, functors which preserve finite limits are **left exact** and dually, functors which preserve finite colimits are **right exact**.

Example. We can construct a large class of functors as follows. Given a category A and any object $a : A$, a **representable functor** is a **hom-functor**: $Hom(a, -) : A \to \texttt{Set}$ which maps objects $b : A$ to the hom-set $\texttt{Hom}_A(a, b)$. We say that $F : A \to \texttt{Set}$ is **representable** if there exists an object $u : A$ such that there is a natural isomorphism $\alpha : \texttt{Hom}_A(u, -) \cong F$. Somewhat similarly, a presheaf F is **representable** if there exists an object $u : A$ such that there is a natural transformation $\alpha : F \cong \texttt{Hom}_A(-, u)$.[14]

Example. Given a category A, one very important category is the category of pre-sheaves, denoted $\texttt{Set}^{A^{op}}$ or more conveniently, $\texttt{PSh(A)}$. The objects are simply the contravariant functors $F : A \to \texttt{Set}$, and the morphisms are the natural transformations between $\alpha : F \to G$. Revisiting the earlier example of reflective graphs, we can define directed graphs with presheaves as follows:

Define category A as having two objects V, E and as the non-identity morphisms, the pair of maps $s, t : V \to E$. For a pre-sheaf G, we identify $G(V)$ as the set of vertices, $G(E)$ as the set of edges, and $G(s), G(t) : G(E) \to G(V)$ as the source and target maps respectively. In effect, if we abuse our notation a little, given $a, b : G(V)$, we can define an element $e : G(E)$ as the pair $\{a, b\}$ such that $e :\equiv \{a, b\}$, with $G(s)(\{a, b\}) = a$ and $G(t)(\{a, b\}) = b$.

We can recover a directed category of reflexive graphs by considering the presheaves G such that for each $a : G(V)$, there is an edge $e_a : G(E)$ such that $G(s)(e_a) = G(t)(e_a)$. This is intuitively pleasing, as it suggests that a self-directed edge at a point a is merely a path between the source and the target functions on the point. Moreover, when working with reflexive graphs, there is no harm (and indeed, it is quite useful), to note that for each pre-sheaf G there is a mapping $e_G : G(V) \to G(E)$ which maps each vertex to the associated self-directed edge, such that $G(s) \circ e_G = G(t) \circ e_G = 1_{G(V)}$. In fact, this is precisely the property[15] we shall use to identify $\texttt{RGph}$ as a subcategory of $\texttt{PSh(A)}$, where A is as above. It is also interesting enough to remark upon the fact that this entails that graph homomorphisms are simply natural transformations on this underlying category A.

Furthermore, we'll abandon the notation of source and target maps as the image of presheaf on the morphisms of A, and just denote s, t, as the meaning of this is clear.

1.2. Adjoints: Functors, Monads, and Modalities

Definition. We say two functors F, G are **adjoint** if for all $a : A, b : B$, $\texttt{Hom}_A(a, G(b)) \cong \texttt{Hom}_B(F(a), b)$. We denote this by $F \dashv G$. Adjoints arise when there are natural transformations $\eta : 1_A \to G \circ F$ and $\varepsilon : F \circ G \to 1_B$ called the **unit** and **co-unit** respectively which satisfy the zig-zig identities:[16]

$$(\varepsilon \diamond_l F) \star (F \diamond_r \eta) = 1_F \text{ and } (G \diamond_r \varepsilon) \star (\eta \diamond_l G) = 1_G$$

Example. Let A be a category and S be a full subcategory of A. If under the inclusion functor $\iota : S \to A$, ι has a left adjoint, $\rho : A \to S$, then S is **reflective subcategory**. Dually, if $\iota \dashv \Xi$, then S is a

[14] Indeed, this is precisely the preoccupation of the Yoneda embedding, which is a functor $Y : C \to \texttt{PSh(C)}$, where $\texttt{PSh(C)}$ is the category of presheaves of C. We will elaborate on this further, but it is worth noting that morphisms of this category are simply natural transformations.

[15] That is, when mapping from the category of reflexive graphs to $\texttt{PSh(A)}$, we have a full and faithful mapping to the collection of presheaves G for which there is a set function $e_G : G(V) \to G(E)$ such that $G(s) \circ e_G = G(t) \circ e_G = 1_{G(V)}$

[16] It should be noted that this is simply a path between the identity 2-morphism and the composition of left and right whiskerings. Sadly, this is not written in the incredibly intuitive left to right manner, but in the fashion of right to left so as to mirror composition.

coreflective subcategory. Please notice that nothing precludes a subcategory from being both reflective and coreflective![17]

Adjoints have some pretty nifty properties. Let $F \dashv G$. Then F preserves all colimits of A and G preserves all limits of D. If F is a left exact functor, then right adjoint G preserves all finite limits and colimits. Hence F is exact.

Example. Univalence implies that the type $\texttt{Prop} :\equiv \sum_{X:\mathcal{U}} \texttt{isProp}(X)$ classifies monomorphisms. With propositional resizing, we can find a small version of $\texttt{Prop}$.

Definition. Let A, B be categories which have finite limits, are cartesian closed and have a **subobject classifier**, and let $f : A \to B$. We say f is a **geometric morphism** if there is a pair of functors (f^*, f_*) of the form $A \underset{f_*}{\overset{f^*}{\leftrightarrows}} B$ such that f^* is left exact and $f^* \dashv f_*$.

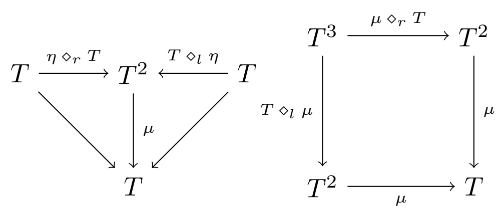
Example. Consider the unit type $\mathbf{1}$ and an arbitrary category A. Clearly, $G : A \to \mathbf{1}$ is a unique functor. If $F \dashv G$, then for any $a : A$, we find $\texttt{Hom}_A(F(\star), a) \cong \texttt{Hom}_{\mathbf{1}}(\star, G(a))$, since only one map exists from $\star \to G(a)$

Example. Adjoints need not come in pairs. Recalling from our earlier examples, the forgetful functor $U : \texttt{Top} \to \texttt{Set}$, and functors $F, G : \texttt{Set} \to \texttt{Top}$, which respectively takes a set to its discrete and indiscrete topology. We find that $F \dashv U \dashv G$, which is an example of an **adjoint triple**. Because F, G, U do not 'resize' the respective hom-sets, it is a rather straightforward exercise to confirm that these are indeed adjoints!

Definition. A **monad** in a category A is a triple given by

(1) an endofunctor T;
(2) a natural transformation $\eta : 1_A \to T$ called the **unit of T**;
(3) a natural transformation $\mu : T \circ T \to T$ called the **multiplication**.

Moreover, these natural transformations satisfy:[18]



Definition. A **comonad** on a category A is a monad on its dual category A^{op}. In particular, there is a natural transformation $\varepsilon : T \to 1_A$, called the **co-unit of T** and a natural transformation $\Delta : T \to T \circ T$ called the **co-multiplication**, which satisfy the appropriate commuting diagrams.[19]

Definition. An **adjoint cylinder** is an adjoint triple $F \dashv G \dashv H$ such that the adjoint pair on of the two sides consists of identity functors and the other side consists of an idempotent monad or comonad.

Every adjoint triple induces an adjoint pair of endofunctors that underlie a monad induced by adjunction.[20] Specifically, we can see $G \circ F \dashv G \circ H$ where $G \circ F$ underlies a monad induced by the adjunction $F \dashv G$ and $G \circ H$ underlies a comonad induced by the adjunction $G \dashv H$.

Definition. A monad (a, T, η, μ) is **adjoint** to a comonad $(a, G, \varepsilon, \Delta)$ if $T \dashv G$ and Δ, ε are the adjoints to μ and η respectively.

Modality is a fairly abstruse philosophical concept. Part of the difficulty with reading philosophers who are seriously preoccupied with metaphysics (such as Hegel or Heidegger) is that absent some formalized language, it is hard to get a sense that a concept like a **mode of being** described by their informal ontologies

[17] In fact, as we shall soon see, there are certainly many intriguing sub categories where $\rho \dashv \iota \dashv \Xi$.

[18] Reader, be forewarned, T^n for $n \in \mathbb{Z}^+$ does not mean the product of T, but rather the the standard functional power, i.e $T^2 \equiv T \circ T$, $T^3 = T \circ T \circ T$, and so forth.

[19] The reader at this point should be able to figure out what these diagrams are by identifying the appropriate diagrams, reversing the arrows and properly relabeling them.

[20] Hopefully, the factorization system is starting to become apparent.

is anything more than meaningless theological warbling about *soul-stuff* that the writer is certain, absolutely certain, takes priority over formal epistemology.[21] Absent Lawvere's profoundly insightful identification of an adjoint structure to describe the *unity of being and nothing*, reading Hegel's description[22] pretty much leaves one with the impression that natural language is probably the worst medium in which one can reason, and moreover, that there is not even a kernel of a good idea lurking in his exposition.[23] However, courtesy of Mike Shulman, we can think of modalities as a higher inductive type, as elaborated in [19], although this is not necessary when first encountering cohesion.

Roughly speaking, a modality is a function $M : \mathcal{U} \to \texttt{Prop}$ that tells us for every type A whether A has a given property M. If M is a modality, then for every type A, there is anther type $\bigcirc(A)$ such that $M(\bigcirc(A))$ holds. In many cases, particularly the several concerning cohesion that this paper examines, modalities are identified with monads or comonads on either a subuniverse of propositions, or on the underlying type universe. In particular, these would be idempotent monads, which in turn means endofunctors. This intuition checks out.

Example. $\texttt{isSet}\colon \mathcal{U} \to \texttt{Prop}$ is a modality for which the $\bigcirc$ is given by the set truncation $\|A\|_0$. Once we have truncated to $\|A\|_0$, further truncation does nothing but preserve our set truncation. Similarly $\texttt{isProp}$ may be thought of as a modality; indeed, in homotopy type theory, the most frequently encountered modalities are the $n-$truncations, for $n \in \mathbb{Z}_{\geq -2}$.

Moreover, we can think of modalities as **stable factorization systems**. That is

Definition. Let A be a precategory. Let (E, M) form two classes of morphisms. If (E, M) form two classes in A such that

(1) for every $f : \texttt{Hom}_A(a, b)$ factors into $f = r \circ l$, with $l : E$ and $r : M$ such that these factorizations are unique up to isomorphism;
(2) E, M contain all isomorphisms;
(3) and are closed under composition;
(4) any $l : E$ and $r : M$, where $l : a \to b$ and $r : c \to d$, and (u, v) are morphisms $u : a \to c$ and $v : b \to d$ such that $r \circ u = v \circ l$, there is a morphism $\gamma : b \to c$ satisfying the lifting problem:

$$\begin{array}{ccc} a & \xrightarrow{u} & c \\ {\scriptstyle l}\downarrow & \nearrow^{\exists\gamma} & \downarrow{\scriptstyle r} \\ b & \xrightarrow[v]{} & d \end{array}$$

If the lift γ is unique, then we say f is orthogonal to g, which we denote by $f \perp g$, and (E, M) forms an **orthogonal factorization system**, as for any $f : E$, $f \perp g$ for any $g : M$. Furthermore, if (E, M) is stable under pullbacks,[24] then (E, M) forms a **stable factorization system**.

In a sense, the modality arises by factoring the underlying endofunctor.

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[21]This is mostly directed at Heidegger.

[22]To quote Hegel's *Science of Logic* [7],

"Pure Being and pure nothing are, therefore, the same. What is the truth is neither being nor nothing, but that being – does not pass over but has passed over – into nothing, and nothing into being. But it is equally true that they are not undistinguished from each other, that, on the contrary, they are not the same, that they are absolutely distinct, and yet that they are unseparated and inseparable and the each immediately vanishes in its opposite. Their truth, is therefore, this movement of the immediate vanishing of the one into the other: *becoming*, a movement in which both are distinguished, but by a difference which has equally immediately resolved itself." What will shortly be shown is that this can be made to comport with the notion of *cohesion*, which is a composition of modalities. What cannot be shown is that this is what Hegel actually had in mind.

[23]For further evidence of this, just try reading through Hegel's writings on the spurious infinite without thanking Cantor for so elegantly proving $\aleph_0 < \mathfrak{c}$.

[24]A category A is said to have pullbacks if for any pair of morphisms, $f : a \to c$ and $g : b \to c$, there exists an object $p : A$ and a pair of morphisms $p_1 : p \to a, p_2 : p \to a$ such that $f \circ p_1 = g \circ p_2$ and with the property such that for any other triple

Example. In a topos, the class E consists of epimorphisms, while M consists of the monomorphisms[25] . Moreover this is stable under pullback (simply consider that the UMP of epimorphism).

Specifically, in the category `Set`, we recognize these as the class of surjections and injections respectively, and in the category `Grp` as surjective and injective homomorphisms. However, in `CRing`, the class E consists of the localizations and M is the class of conservative[26] homomorphisms.

1.3. Some Rudimentary Topos Theory Geometric morphisms are morphisms over a very special kind of category called a **topos**. Fittingly, given the name of Johnstone's tome on the subject[27], provided below are three equivalent definitions of an elementary topos.

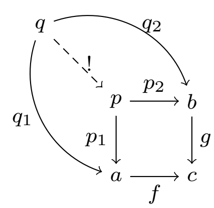
Definition. A topos is a category $\mathcal{E}$ which

(1) has finite limits;[28]
(2) has an object $\Omega : \mathcal{E}$, called the **subobject classifier**, which is a pointed object classifying monomorphisms, along with a function P which assigns to each object $a : \mathcal{E}$ an object $P(a) : \mathcal{E}$, where $P(a)$ is called the **power object of a**, which can be thought of as a generalization of the power set construction in set theory;
(3) the functors $\mathrm{Sub}_{\mathcal{E}}$ and $\mathtt{Hom}_{\mathcal{E}}(b \times -, \Omega)$ such that for each object $a : \mathcal{E}$, we have two natural isomorphisms $\mathrm{Sub}_{\mathcal{E}} a \cong \mathtt{Hom}_{\mathcal{E}}(a, \Omega)$ and $\mathtt{Hom}_{\mathcal{E}}(b \times a, \Omega) \cong \mathtt{Hom}_{\mathcal{E}}(a, Pb)$.

Definition. Alternatively, we can define a topos as being a category A which:

(1) has finite limits;

(q, q_1, q_2)



commutes. This is the universal mapping property of pullbacks. Dually, we can define pushouts. The interested reader is invited to test these definitions out with what they know about products and coproducts.

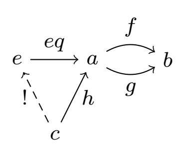
[25] In case the reader is unfamiliar with these terms, in a category A, a morphism $f : a \to b$ is said to be **epic** when $\prod_{c:A} \prod_{g,h:\mathtt{Hom}_A(c,a)} ((g \circ f) = (h \circ f)) \to (g = h)$; if you're not comfortable with framing of propositions as types, if for all g, h, $g \circ f = h \circ f$ implies $g = h$, then f is an **epimorphism**. A **monomorphism** is an epimorphism in the dual category, and so it is left to the curious reader to come up with either a type-theoretic, or conventional description of the universal property of monomorphisms.

[26] As a reminder, we say a ring homomorphism $\varphi : A \to B$ inverts an element $f : A$ if $\varphi(f)$ is invertible in B and we say φ is **conservative** if every element of $f : A$ which is inverted by φ is invertible in A. For any $S \subseteq A$, there is a commutative ring $S^{-1}A$ with ring homomorphisms $\lambda : A \to S^{-1}A$ which universally invert every element of S. This means that for any ring homomorphism φ which inverts every element of S, there exists a unique $\psi : S^{-1}A \to B$ such that $\psi \circ \lambda = \varphi$.

[27] Sketches of an Elephant,[10]

[28] In effect, this entails that A has equalizers and co-equalizers, along with finite products and coproducts. This necessitates the following definitions for the uninitiated:

Definition. A category A is said to have an **equalizer** for $a, b : A$, and a pair of parallel morphisms $a \overset{f}{\underset{g}{\rightrightarrows}} b$, if it has an object $e : A$ and a morphism $eq : e \to a$ which are a limit to the parallel pair of morphisms, i.e.



commutes. Moreover, one can show that all equalizers are monic. One can also describe the equalizer as the dependent sum over the dependent equality type $e \simeq \sum_{a:A} (f(a) = g(a))$, as described in[3]. One can dually define **co-equalizers**.

Definition. A category A has **finite products** if for every finite index category $\mathcal{I}$ with objects X_i in A , $\prod_i X_i$ is an object in the category. Dually, a category A has **finite co-products** if for every finite index category $\mathcal{I}$, with objects X_i in A, $\coprod_i X_i$ is an object in this category.

(2) is cartesian closed;[29]
(3) has a subobject classifier Ω, which is an object Ω and an arrow $true : 1 \to \Omega$ such that for each monic $f : a \to b$, there is one and only one $\chi_f : d \to \Omega$ such that

$$\begin{array}{ccc} a & \overset{f}{\rightarrowtail} & d \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

is a pullback square.

Definition. An **elementary topos** is a category $\mathcal{E}$ such that
(1) $\mathcal{E}$ has a terminal object and pullbacks;
(2) $\mathcal{E}$ has an initial object and pushouts;
(3) $\mathcal{E}$ has exponentiation;
(4) $\mathcal{E}$ has a subobject classifier.

The interested reader can verify these are equivalent[30] definitions. Throughout this paper, this third definition will be the one considered.

Example. The canonical example of a topos is the category of sets, where the sub-object classifier consists of the characteristic functions and $\Omega = \{0, 1\}$ such that

$$\begin{array}{ccc} a & \overset{f}{\rightarrowtail} & d \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

commutes.

We also recognize that $false : 1 \to \Omega$ is defined by $false(*) = 0$. Now, since $false : 1 \to \Omega$ is a unique arrow, we can recognize this as the characteristic function defining a subobject classifier for some monic arrow. In this case, the characteristic function is of the unique map from $\emptyset \to 1$, giving us the following pullback square:

$$\begin{array}{ccc} \emptyset & \overset{!}{\rightarrowtail} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle false} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

which, in turn, can be used to describe the false maps for any topos. Thus we can say $false = \chi_{1^0}$, where 1^0 indicates the unique map from the initial object to the terminal object.

Furthermore, working within the topos of sets, we can recognize that $f : a \rightarrowtail d$ are the inclusion maps $a \hookrightarrow d$, as these are simply abstract sets. Moreover, in the topos `Set`, the power object $P(a)$ or rather Ω^a is just the powerset relation 2^a we all know and love, as we can identify $\mathfrak{P}(a) \cong 2^a$ when working with sets.[31]

[29]A cartesian closed category is a category A with a terminal object $1 : A$, finite products (i.e, for all $a, b : A$, $a \times b : A$), and for any two objects $a, b : A$, there is an exponential object b^a and a morphism $\texttt{eval} : (a \times b^a) \to b$ such that for any object c and a morphism $f : (c \times a) \to b$, there is a unique morphism $\hat{f} : c \to b^a$ such that

$$\begin{array}{c} c \\ {\scriptstyle \hat{f}}\,\vdots\downarrow \\ b^a \end{array} \qquad \begin{array}{ccc} c \times a & & \\ {\scriptstyle \hat{f} \times 1_a}\,\vdots\downarrow & \searrow^{f} & \\ b^a \times a & \underset{\texttt{eval}}{\longrightarrow} & b \end{array}$$

The exponential construction is universal construction. If intuition is lacking about what b^a and `eval` really consist of, consider that b^a may be though of the generalized collection of morphisms $g : a \to b$, and $\texttt{eval}(g, a) = g(a)$. The reader is encouraged to consult [9] to see how the exponential object is related to a power object.
[30]Curiously enough, the first two definition come from Moerdijk, the first from [15] and the second from [14]. The final definition is courtesy of [9]
[31]If $a, b : \texttt{Set}$, and $\chi_a = \chi_b$, where χ_a and χ_b are the standard characteristic functions of sets a,b that identify whether a point in a set c is included in a (b, resp), then we have $a = b$, as they agree on all points, namely $\chi_a(a) = \chi_b(a) = \chi_a(b) = \chi_b(b) = \{1\}$.

In homotopy type theory, if a : $\mathtt{Set}$ and if for the type family $P : a \to \mathcal{U}$, for each $x : a$, $P(x)$ is regarded as a proposition, then we can refer to P as a membership predicate[32] and identify *subsets* of $b \subseteq a$, with the following useful dictionary between set predicates and dependent pair types:

Rather curiously,[33] within the type theoretic construction, where $b : \mathtt{Set}$, every $x : b$ is indistinguishable as sets in homotopy type theory are not the sets of ZFC, but rather, are the abstract sets we're playing with in the topos of sets. From this example we can see one thing about sets: the logic of $\mathtt{Set}$ is an account of set membership. This structure is rather brutal, as it ignores certain subtleties which may be of interest. Luckily, we have other topoi for that!

Remark. In category theory, if a category A has a terminal object 1, we can define the **elements** of other objects $a : A$ as the class of arrows $1 \to a$. In the case of a topos $\mathcal{E}$, the class of arrows $1 \to \Omega$ are the **truth-values** of $\mathcal{E}$.

In fact, one can think of a topos as being a generalization of the category of sets, in so much as one can do mathematics within different topoi. The following are some interesting species of topoi, as they relate to *cohesion*, which bear mentioning, but will not be further developed in this paper.

1.3.1. Some Interesting Topoi For Studying Smooth Spaces

Example. Ringed topoi emerge rather naturally from the fact that a topos is a cartesian monoidal category.[34] In this case, given a topos $\mathcal{E}$, one can define internally[35] the notion of a (commutative) unital ring. Specifically, these are pairs[36] $(\mathcal{E}, \mathcal{O})$ where $\mathcal{O}$ is a distinguished unital ring object internal to the topos $\mathcal{E}$.

Example. Lined topoi $(\mathcal{E}, R)$ are ringed topoi equipped with both the usual internal ring object $\mathcal{O}$ and a choice of an internal commutative algebra object R over $\mathcal{O}$, called the **line object**.

One interesting example of a lined topos are the sheaves[37] of cartesian spaces, denoted $\mathtt{Sh}(\mathtt{CartSp}_{\mathtt{smooth}})$ whose lined object is the interval[38] object $1 \coprod 1 \to \mathbb{R}$. Despite the rather expansive expository nature of the paper, verifying that $\mathtt{Sh}(\mathtt{CartSp}_{\mathtt{Smooth}})$ is a topos will probably further test the reader's patience. Suffice to say it is an instance of a Grothendieck topos,[39] as $\mathtt{CartSp}_{\mathtt{Smooth}}$ can be made into a small site with a little leg work.[40]

[32]Indeed, it is worth recalling that type families are fibrations, in so much as given a type $A : \mathcal{U}$, then the type family $P : A \to \mathcal{U}$ is a **fibration** with base space A and each $P(x)$ is a **fibre** over x, with the dependent pair $\sum_{x:A} P(x)$ characterizing the **total space of the fibration**. In the case of propositional membership, we have the familiar predicate logic of First Order Logic, but within the internal logic of other decidedly less quotidian topoi, we often have a typed higher order logic with higher order predicates. One could almost say that this motivates the hunt for a means of modeling homotopy type theory in any elementary $(\infty, 1)$-topos (although that would very reductive).

[33]Although curious, this is not shocking at all, and can be considered an instance of the Mengen/Kardinalen paradox that motivated Lawvere to study cohesive sets in the first place!

[34]See Section III for a full comment on what a monoidal category is. In this case, the categorical product gives the monoidal structure, and the terminal object acts as the unit.

[35]There are two important ways to generalize a category: internalization and enrichment. The gist of an internal category is that within a category A with enough pullbacks, one can construct *another category* if there is an object $V : A$ and an object $E : A$, together with source and target morphisms $s, t : E \to V$, an identity assigning morphism $e : V \to E$ and a composition morphism $c : E \times_V E \to E$ satisfying the usual category laws.

[36]Looks an awful like a ringed space, no?

[37]Sheaves are merely pre-sheafs with additional topological structure which tracks the local data of an open set. The two additional requirements are the *locality* and *gluing* requirements: (locality) given an open covering (U_i) of an open set U, and $s, t : F(U)$ such that $s|_{U_i} = t|_{U_i}$ for each i, then $s = t$; (gluing) If for each pair U_i, U_j in the open cover, there two respective sections s_i, s_j which agree on overlaps (i.e $s_i|_{U_i \cap U_j} \cap s_j|_{U_i \cap U_j}$, there is a third section $s \in F(U)$ such that $s|U_i = s_i$ for each i).

[38]In categories with finite limits, such as topoi, a lined object is the copairing of maps $f, g : 1 \to I$, where 1 is a terminal object. This is to say, $[f, g] : 1 \coprod 1 \to I$ is an interval object. In general, it is good practice to associate the interval object to the unit interval.

[39]These are topoi that are equivalent to the categories of sheaves on a small site.

[40]This is tantamount to proving that every paracompact smooth manifold admits a good open cover.

Set Theory	HoTT
$\{x \in a \mid P(x)\}$	$\sum_{x:A} P(x)$

Example. A **smooth topos** is a lined topos where each functor $(-)^{\mathrm{Spec} W} : \mathcal{E} \to \mathcal{E}$ defined by an **R-Weil algebra**[41] W has a right adjoint, known as the **amazing right adjoint**, and the canonical $W \to R^{\mathrm{Spec}(W)}$ is an isomorphism. Effectively this means that each $R-$Weil algebra is infinitesimal and satisfies the Kock-Lawvere axiom[42].

1.3.2. Return of the Reflexive Graphs: The Triumph of the Topos

Claim 1. *The category* `RGph` *is also a topos.*

Proof. We proceed by proving the following claims:

Claim 2. `RGph` *has initial and terminal objects.*

Proof. It is easy to verify that the empty graph is an initial object, as the empty set is an initial object in the category of sets. Given the dependence of f_2 on f_1 in the pair of morphisms (f_1, f_2) which identify a graph morphism, it is clear that a map from the empty graph to any graph $G(V), G(E)$ is unique both by the uniqueness of $f_1(\emptyset) = G(V)$, and by the uniqueness of f_2 up to isomorphism which identifies all edges with their respective pairs $\{a, b\}$ in $G(E)$. If there were two such maps f_2, g_2, which identified the edges $G(E)$, it is immediately clear that the two are isomorphic, as $f_1 = g_1$ is immediate.

Similarly, since singletons are terminal objects in the category of sets, the graph consisting of one vertex and its self directed edge is a terminal object in `RGph`. □

Claim 3. `RGph` *has pullbacks and pushouts.*

Proof. This claim is equivalent[43] to proving `RGph` has finite products and equalizers, along with finite co-products and co-equalizers.

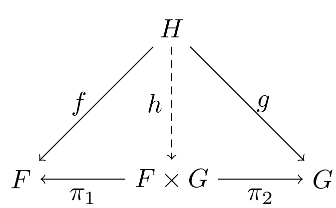
So it suffices to show that `RGph` has finite products and coproducts, and then showing it has equalizers and co-equalizers.

1.3.3. Products First, we can define the binary categorical product (which is not the same as the cartesian product) of graphs as follows:
$(G \times H)(V) = G(V) \times H(V)$ and the pair of pairs $\{\{a, b\}, \{c, d\}\}$ is identified with an edge $e_{GH} : (G \times H)(E)$ if $\{a, c\}$ is identified with an edge $e_G : G(E)$ and $\{b, d\}$ with an edge $e_H : H(E)$. In particular if $s(e_G) = a, t(e_G) = c, s(e_H) = b$ and $t(e_H) = d$, then $s(e_{GH}) = \{a, b\}$, which we'll denote by e^s_{GH}, and $t(e) = \{c, d\}$ which we'll denote by e^t_{GH}.

Importantly, when looking at the well known cartesian products for sets, the projection maps act on the product of the vertex sets as expected; we can use the standard projection maps. What we need to to is define the product maps on the edge product.

We define the product maps π_1 and π_2 on edges e, where $e = \{e^s, e^t\}$, by $\pi_1(e) :\equiv \{s(e^s), s(e^t)\}$ and $\pi_2(e) :\equiv \{t(e^s), t(e^t)\}$, which are equal to e_1 and e_2 respectively. In effect, we have "lost" our edge information, although we still have a "point"![44]

We can check that this indeed satisfies the limit notion of a categorical product, i.e. given H with graph homomorphisms $f : H \to F$ and $g : H \to G$, there is a unique map $h : H \to F \times G$ such that



commutes.

To see that there exists such a map, define $f \times g : H \to F \times G$ as the following pair of products maps $(f_1 \times g_1, f_2 \times g_2)$:

[41]An R-**Weil algebra** is an R-algebra of the form $W = R \oplus J$, where J is an R-finite dimensional nilpotent ideal.
[42]Very briefly, this is the requirement that this topos requires every morphism from an infinitesimal interval $D \subset R$ into R is linear and can be extended uniquely to a linear map $R \to R$.
[43]Technically, the result is category A has products and equalizers if and only if A has pullbacks and terminal objects; one can dually obtain the other result.
[44]Which in fact is an edge, and so for the sake of completing this half formed pun, we're still edgy!

- $f_1 \times g_1$ is simply the product map of the product of the sets of vertices, i.e. $f_1 \times g_1 : H(V) \to F(V) \times G(V)$, where in standard set notation
$$(f_1 \times g_1)(H(V)) := \{(f_1(x), g_1(x)) \in F(V) \times G(V) \mid x \in H(V)\}$$
- $f_2 \times g_2$ requires a little more delicacy.
Since both f, g are graph homomorphisms, we have f_2, g_2 are maps from $H(E)$ to $F(E)$ and $G(E)$ respectively. In particular, let $e_H : H(E)$, and denote $f(e_H) = e_F$ and $g(e_H) = e_G$. Now define
$$(f_2 \times g_2)(e_H) :\equiv \{\{s(f_2(e_H)), s(g_2(e_H))\}, \{t(f_2(e_H)), t(g_2(e_H))\}\} \equiv e_{FG}$$
for some $e_{FG} : (F \times G)(E)$. Very quickly, with this definition see that:
$$\begin{aligned}
\pi_1((f_2 \times g_2)(e_H)) &= \pi_1(e_{FG}) \\
&= \{s(e^s_{FG}), s(e^t_{FG})\} \\
&= \{s(\{s(f_2(e_H)), s(g_2(e_H))\}), s(\{t(f_2(e_H)), t(g_2(e_H))\})\} \\
&= \{s(f_2(e_H)), t(f_2(e_H))\} \\
&= f_2(e_H) \\
&= e_F
\end{aligned}$$
A similar result follows for π_2.
To quickly check that this construction is unique, suppose that there is some $h : H \to F \times G$, such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ as above. Immediately, we find that $h_1 = f_1 \times g_1$, given the universal properties of products over sets. Similarly, for any edge $e_H : H(E)$, we find that $\pi_1(h_2(e_H)) = f_2(e_H) = \{s(f_2(e_H)), t(f_2(e_H))\}$ and $\pi_2(h_2(e_H)) = g_2(e_H) = \{s(g_2(e_H)), t(g_2(e_H))\}$, but as this defines an edge in $F \times G$, we find that for each $e_H : H(E)$, $h_2(e_H) = (f_2 \times g_2)(e_H)$, whence $h_2 = f_2 \times g_2$.

Hence binary products exists. Using this construction, one can then make graphs $(F \times G) \times H$ and so forth, and moreover, verify that $(F \times G) \times H \cong F \times (G \times H)$, rather straightforwardly using the universal mapping property of products.

1.3.4. Coproducts Similarly, $G \coprod H$ is a graph defined as follows:
$G(V) \coprod H(V)$ is the standard coproduct of sets, and $\{a, b\}$ is identified with an edge in $(G \coprod H)(E)$ if $\{a, b\}$ is identified with an edge in $G(E)$ or in $H(E)$. The reader is invited to check that this definition is indeed a categorical coproduct.

1.3.5. Equalizers and Co-equalizers Now to check that we have equalizers and co-equalizers.

Suppose that we have graph morphisms $f, g : G \to H$. An equalizer eq is simply the inclusion map $i : E \hookrightarrow G$ where $E \subset G$ is the induced subgraph on the set of vertices such that $E(V) = \{a : G(V) \mid f(a) = g(a)\}$. Similarly, the co-equalizer is a projection map $p : H \to C$ where C is the graph of the coset of vertices $H(V)/\sim$ where $\sim$ is an equivalence relation on $H(V)$ given by $f(a) = g(a)$ for some $a : G(V)$. Edges in C are given by the induced pairing of equivalence classes $\{[b], [b']\}$ when $\{b, b'\}$ has a corresponding edge in $H(E)$. □

Claim 4. *$\mathtt{RGph}$ has exponential objects.*

Proof. Given graphs G, H, the exponential graph H^G is a graph whose set of vertices are the vertex morphisms of the graph homomorphism $f : G \to H$, i.e. $H^G(V) = \{f_1 : G(V) \to H(V)\}$, with edges $\{f_1, g_1\}$ in $H^G(E)$ identified as follows: for each pair $\{a, b\}$ with an edge in G(E), $\{f_1(a), g_1(b)\}$ is identified with an edge in $H^G(E)$.

Moreover, there for graphs F, G, H, there is a natural isomorphism $\alpha : \mathtt{Hom}(G \times F, H) \cong \mathtt{Hom}(F, H^G)$ given by $(\alpha(f)(b))(a) = f(b, a)$ for all $f : \mathtt{Hom}(G \times F, H), a : F(V), b : G(V)$. We can verify this be letting $f : F \times G \to H$ be an inhabitant of the appropriate hom set, and supposing that $\{a, a'\}$ is an edge in $F(E)$ and $\{b, b'\}$ is an edge in $G(E)$. We then find $\alpha(f)(a)(b) = f(a, b)$ and $\alpha(f)(a')(b') = f(a', b')$, which gives us an edge in H, since f is a graph homomorphism.

We check naturality rather easily by supposing that $f : F \to F'$ and $g : H \to H'$ are graph homomorphisms, and letting $\varphi : \mathtt{Hom}(F' \times G, H)$. Then we have $(\alpha(1_G \times f, g))(\varphi)(a)(b) = (1_G \times f, g)(\varphi)(a, b) =$

$g(\varphi(b, f(a)))$ and $((f, g^G)(\alpha))(\varphi)(a)(b) = g(\alpha(\varphi))(f(a)(b)) = g(\varphi(f(a), b))$. One can easily check this gives rise to the following commuting diagram:

$$\begin{array}{ccc} \mathtt{Hom}(G \times F, H) & \xrightarrow{\alpha} & \mathtt{Hom}(F, H^G) \\ {\scriptstyle (1_G \times f, g)}\downarrow & & \downarrow{\scriptstyle (f, g^G)} \\ \mathtt{Hom}(G \times F', H') & \xrightarrow[\alpha]{} & \mathtt{Hom}(F', (H')^G) \end{array}$$

Since the diagram commutes, we find that α is a natural isomorphism, from which $\mathtt{Hom}(G \times F, H) \cong \mathtt{Hom}(F, H^G)$, and thus we find that $\mathtt{RGph}$ indeed have exponentials. □

Claim 5. *$\mathtt{RGph}$ has a subobject classifier.*

Proof. Before beginning, it suffices to note that for any monic graph homomorphism $m : H \rightarrowtail G$, we can identify H up to isomorphism with a subgraph $S \hookrightarrow G$, moreover, since we're working with *abstract* graphs and for the sake of less cluttered notation, we will consider all sources of monic maps as subgraphs.

Immediately, we recognize that our subobject classifier will need to be a graph Ω. Thus $\Omega(V)$, as a set, has the set subobject classifier $\{0, 1\}$. Thinking back to the presheaf category of reflexive graphs, we can denote $\Omega(V) = \{0_V, 1_V\}$. In the case where $a : G(V)$ but a is not in $S(V)$, $\chi_m(a) = 0_V$, while if $a : G(V)$ and $a : S(V)$, then $\chi_m(a) = 1_V$.

However, the internal logic for the category of reflexive graphs is decidedly not Boolean. One only needs to consider the possibilities regarding edges to see why this is the case, so allowing some abuse of notation

(1) Suppose $\{a, b\} : S(E)$, then we can assign $\chi_m(\{a, b\}) = 1_E$;
(2) Suppose $\{a, b\} : G(E)$, but not in $S(E)$, we have the following four distinct possibilities:
    (i) Further suppose $\chi_m(a) = 0_V$ and $\chi_m(b) = 0_V$, then in this case, we assign $\chi_m(\{a, b\}) := 0_E$;
    (ii) Further suppose $\chi_m(a) = 1_V$, and $\chi_m(b) = 0_V$, then in this case, we assign $\chi_m(\{a, b\}) := s$, indicating the source is 'valid'[45]
    (iii) Further suppose $\chi_m(a) = 0_V$, and $\chi_m(b) = 1_V$, then in this case, we assign $\chi_m(\{a, b\}) := t$, indicating the target is 'valid';
    (iv) Further suppose $\chi_m(a) = 1_V$ and $\chi_m(b) = 1_V$, then in this case, we assign $\chi_m(\{a, b\}) := (s, t)$, indicating that both the source and target are valid, but the edge itself is not present in the graph.

In effect, we can identify Ω with $\Omega(E) = \{0_E, s, t, (s, t), 1_E\}$, which is partially ordered by $0_E < s, t < (s, t) < 1_E$. □

With these claims, we can verify that $\mathtt{RGph}$ is indeed a topos. We'll soon see that is also happens to be a *cohesive* topos. □

Remark. It is also a rather interesting, if not altogether trivial observation, to notice that each self directed edge is identified with 1_E, in effect, identifying each vertex as a "true" object. Moreover, the truth values corresponding to the self-directed edges are invariant across all reflexive graphs. Somewhat similarly, in a complete graph, every edge is evaluated as 1_E, and so every possible relation between vertices is "true". Of course, things that are tautologically true are not necessarily those things which we find interesting!

Definition. A **local geometric morphism** is an adjoint triple $F^* \dashv F_* \dashv F^! : B \to A$ such that for all $a, b : B$, F^* is such that

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[45] It bears commenting that one of the most powerful features of topos theory is that one can work with the internal logic of that category, most notably by treating sub objects as (higher order) predicates, and forming a poset of the truth values that can be made into a Heyting algebra. In this case, this non-boolean internal logic conveys more information about structure than would otherwise be permitted if we worked strictly in $\mathtt{Set}$ and with the standard $\in$ predicate. In this case, the 'validity' really only refers to a commuting diagram and the respective truth value which can be identified rather loosely with the proposition of "is an edge whose source is a vertex in the graph, but whose target is not". Rather than simply collapsing this information as true or false, we have five truth values which can form a Heyting algebra. For the reader curious about the internal logic of a topos, start with [9] and [15]. Truly adventurous readers can further develop the Propositions as Type Perspective, developed in [1][p 55], and draw out the corresponding dictionary between the familiar logical operators and those that emerge in categorical semantics (which would have been provided but for the sake of keeping this paper compact!)

(1) (full) $F^* : \mathtt{Hom}_B(a,b) \twoheadrightarrow \mathtt{Hom}_A(F^*(a), F^*(b))$
(2) (faithful) $F^* : \mathtt{Hom}_B(a,b) \rightarrowtail \mathtt{Hom}_A(F^*(a), F^*(b))$.

Definition. A **local topos** $\mathcal{E}$ is a sheaf topos where the global section geometric morphism $\mathcal{E} \underset{\Gamma}{\overset{Lconst}{\leftrightarrows}} \mathtt{Set}$ has a further right adjoint $\mathtt{coDisc} : \mathtt{Set} \hookrightarrow \mathcal{E}$, i.e. $Lconst \dashv \Gamma \dashv \mathtt{coDisc}$.

Remark. Considering that types are weak $\infty-$groupoids, we can think of an $(\infty,1)-$topos $\mathbf{H}$ as a type with a canonical $(\infty,1)$ geometric morphism to the type of ∞-groupoids. Objects are generalized spaces and the higher homotopies might carry additional structure to the standard example. Indeed, being able to realize homotopy type theory with higher inductive types and the univalence axiom in the internal logic of arbitrary elementary $(\infty,1)-$topoi is an active area of research (see [17] for a nice in).

2. What Is Cohesion, Anyhow?

Very broadly, cohesion is a means of describing how non-trivial structures on mathematical objects arise functorially in a way that relates individual objects to the underlying conceptual whole. An example which is consistently returned to throughout this paper is the relationship between topologies and their underlying sets.

An abstract set X has no structure, although every element might be distinct, they are indistinguishable without additional structure, and with that further predicates. For example, when considering the structure of a topology, we can run the gamut from indiscrete topologies to discrete topologies, with an endofunctor mapping the indiscrete topology to the discrete topology in effect giving an underlying abstract set X an incredible amount of structure (by making every point distinguishable).

The difficulty remains that if you stick to the theoretical motivations, you beg the question what is the use of this formalism. If you stick to applications, you are liable to miss the actual point of the formalism. Any serious pedagogical discussion of calculus for instance is situated on a knife's edge separating exposition of analysis and differential geometry from imparting formulae for manipulating functions to solve well-posed problems. This is the tension between teaching how to identify and create models with how to work within a model, akin to a dilemma in the pedagogy of the culinary arts between teaching how to create recipes and how to follow them – between composition and technique.

Rather than overload a set with additional predicates, we can deal with discrete possessing the (geometric) structures we're interested in for free by working in a system where the objects already possess this structure. For instance, synthetic differential geometry avoids the complexity of classical definitions by treating the objects as already *being* smooth, and from there, proceeding to work within the internal logic of a smooth topos. This example is somewhat evocative, because in such topoi, given the existence of exponentials, we can generalize the notion of a tangent bundle as the exponential object X^D, where D is an infinitesimal interval and X is simply a smooth object of the topos.

However, cohesion extends far beyond smooth geometry. Lawvere, in seeking a proper science for explicating the formal language of dynamics may have preoccupied himself with problems in continuum physics and combinatorics in his development of cohesion, still went about this in a way where such developments are incidental to the underlying theory. Still, working with cohesive categories allows one to develop a *fundamental physics*, which Schreiber and Shulman have made overtures to in [17] with regards to *quantum gauge field theory*, situated in the language of cohesive homotopy type theory.

One can think of an $(\infty,1)-$topos $\mathbf{H}$, whose objects are $\infty-$groupoids (which you are encouraged to think of as Types), as a collection of spaces (again, think Types), and that cohesion on $\mathbf{H}$ is a means of specifying how points are collected together in a logical structure. For instance, open balls in topological spaces or smooth structures are examples of cohesive structures.

Any type admits both discrete cohesion where no distinct points cohere non-trivially, and a codiscrete cohesion, where all points cohere in every possible way admitted by the cohesive structure. In the case of elementary topoi, we are looking at adjoint functors into the category of sets, while with $(\infty,1)$-topoi, we look at at adjoint functors into the category of $\infty-$groupoids. For instance, $\Gamma : \mathbf{H} \to \infty\mathtt{Gpd}$ is a functor which forgets the cohesive structure (which we can identify with the representable functor $\mathtt{Hom}_{\mathbf{H}}(*,-)$, where $*$ is a terminal object), but identifies the underlying object which possessed the structure.

Cohesion is captured by an adjoint triple of modalities which describe the transition within one Type from one mode of being to another; in a sense, it is simply an algebraic way of changing the point of view one takes on a formal object. This adjoint triple of modalities is given as follows:

$$\texttt{modality} \dashv \texttt{comodality} \dashv \texttt{modality} \equiv \int \dashv \flat \dashv \sharp$$

2.1. Cohesive (Higher) Topos Theory For a local $((\infty,1)-)$topos $\mathbf{H}$ equipped with a fully faithful extra right adjoint $\texttt{coDisc}$ to the global section geometric morphism $\texttt{Disc} \dashv \Gamma$, we define the following idempotent monads:

Definition. $\sharp :\equiv \texttt{coDisc} \circ \Gamma$, where the codiscrete objects are the modal types. For example, in $\texttt{Top}$, the full subcategory of indiscrete topologies consists of the codiscrete objects for this modal type.

Definition. $\flat :\equiv \texttt{Disc} \circ \Gamma$, where the discrete objects are the modal types. A discrete objects can be thought of as a free object for the forgetful functor (i.e., it is an object which is in the image of its left adjoint, like a discrete topological space). For instance, the full subcategory of discrete topologies in $\texttt{Top}$ contains the discrete objects of this modal type.

Definition. The "shape" modality $\int$ builds out of an additional left adjoint Π, which preserves finite products, and depending on the context one is working in, Π computes the connected components or the fundamental $\infty-$groupoid. So $\int :\equiv \texttt{Disc} \circ \Pi$.

This shape modality gives us a "points-to-pieces" transformation which takes points to their geometric realization, in the sense that the connected components are "discretized".

So we identify the adjoint 4-tuple of functors with the adjoint triple of modalities:

$$\Pi \dashv \texttt{Disc} \dashv \Gamma \dashv \texttt{coDisc} \equiv \int \dashv \flat \dashv \sharp$$

Example. A **cohesive set** is an adjoint triple of these modalities $\int \dashv \flat \dashv \sharp$.

Putting all of these pieces together, we now have a clear picture of a cohesive topos.

Definition. A cohesive topos is a strongly connected, connected, locally connected, local topos $\mathcal{E}$, such that the global section geometric morphism $\mathcal{E} \to \texttt{Set}$ gives rise to the adjoint 4-tuple

$$\Pi \dashv \texttt{Disc} \dashv \Gamma \dashv \texttt{coDisc} : \texttt{Set} \to \mathcal{E}$$

which can also be identified with the adjoint triple of modalities

$$\int \dashv \flat \dashv \sharp$$

and where the objects $a : \mathcal{E}$ are spaces endowed with some structure of interest, i.e. *cohesion*, $\Gamma(a)$ is the underlying set, $\texttt{coDisc}$ equips sets with the codiscrete cohesion, $\texttt{Disc}$ equips sets with the discrete cohesion, and $\Pi(a)$ is the set of connected components of object a.

In the higher case, instead of a global section geometric morphism to $\texttt{Set}$, $\Gamma : \mathbf{H} \to \infty - \texttt{Gpd}$. However, we do not need to work exclusively with topoi.

2.2. Axiomatic Cohesion Lawvere's motivation for developing cohesive topos theory appears to have developed organically from his interest in developing continuum physics from the perspective of categorial logic, and his desire to see a means of corralling various background models for dynamical mathematical theories[46] in a sufficiently expressive framework which unifies different mathematical categories by their mutual transformation.

Going beyond topoi, Lawvere identifies that the default categories for a "science of cohesion" are extensive categories.[47] From here, he develops the evocative notion of categories that are **quality types** over other categories, from which one recovers the adjoint quadruple with which the reader is now no doubt familiar, where **extensive** and **intensive** qualities on a category of cohesion are characterized by Π and Γ respectively. This language is philosophically evocative because Lawvere manages to also describe the notoriously contentious concepts **form** and **substance** in this manner, where **form** is an extensive quality, and **substance** is an intensive quality. That these seemingly separate and opposed notions characterize *cohesion*

[46]Lawvere says as much in the opening of [11].

[47]These are, rather informally, (not necessarily closed) cartesian categories with finite coproducts that are preserved by pullbacks

is precisely Lawvere's point,[48] as these notions are realized in each object as, in another the philosophically loaded term, **modalities**.[49]

Of course, this exposition begs the question: how are these bold-faced terms defined?

Remark. Although up until now, functors have been capitalized, in keeping with the spirit of type theory, Awodey's approach to category theory, etc, when discussing Lawvere's work, and the rather idiosyncratic approach he takes therein, functors are now denoted by lower case letters. Moreover, these definitions are almost directly from Lawvere [11], but are rewritten in a way that aims to be more accessible.

Definition. Let Q, B be two extensive categories. If functor $q^* : B \to Q$ is full, faithful, and is both reflective and co-reflective by a single functor $q_! = q_*$, then Q is a **quality type** over B.

Definition. Suppose that E and B are cartesian-closed extensive categories equipped with the adjoint quadruple

$$f_! \dashv f^* \dashv f_* \dashv f^! : B \to E$$

with the following properties:
(1) $f_!$ preserves finite products;
(2) $f^!$ is full and faithful;
(3) For all $a : E$ and $b : B$, there is a natural isomorphism

$$f_!(a^{f^*(b)}) = [f_!(a)]^b$$

i.e. $f_!$ preserves B parameterized exponentials;
(4) the canonical map $f_* \to f_!$ in B is an epimorphism (which he also evocatively refers to as the *Nullstellensatz*).

Then E is a **cohesive category relative to** B.

We may very well recognize that this describes the familiar adjoint quadruple $\Pi \dashv \mathtt{Disc} \dashv \Gamma \dashv \mathtt{coDisc}$. In effect, f_* and $f_!$ express the opposition between "points" on one hand, and "pieces" on the other, while $f^!$ and f^* describe the opposition of *pure cohesion* (codiscreteness) and *pure anti-cohesion* (discreteness). Lawvere [11] goes on to point out that these are identical in B, while uniquely placed as full subcategories in E by f^*. Furthermore, the astute reader will notice that this means that a cohesive category is at least a local topos.

Remark. Returning back to topos theory, now with this relative point of view, we can now define a cohesive topos as a topos $\mathcal{E}$ relative to a **base** topos $\mathcal{D}$, if given a geometric morphism[50] $f : \mathcal{E} \to \mathcal{D}$ such that
(1) There is an additional left adjoint $f_! \dashv f^*$ that preserves finite products and terminal objects in which case $\mathcal{E}$ is, in order, a **locally connected, connected, and strongly connected topos**;
(2) $\mathcal{E}$ is a local topos, with an additional right adjoint $f_* \dashv f^!$;

In fact, it is somewhat instructive to consider that a Grothendieck topos $\mathcal{E}$ is **locally connected** if for each $a : \mathcal{E}$, a is a coproduct of connected objects $a_i : \mathcal{E}$ indexed by $I : \mathcal{U}$.

Interestingly enough, given a cohesive category E relative to an extensive base category B, **qualities** are defined not as a category, but as functors that factor $f_!$ and f_*, the "pieces" and "points", through quality types.

Definition. Let E be a cohesive category relative to B with the adjoint 4-tuple $f_1 \dashv f^* \dashv f_* \dashv f^!$. If $h : E \to Q$ is a functor such that
(1) where Q is a quality type A over B with $q_* = q_! : Q \to B$;
(2) h preserves finite co-products;
(3) $q_! \circ h = f_! : E \to B \equiv E \to Q \to B$

then H is an **extensive quality on category E**. If $g_* : E \to B$ is a functor that
(1) preserves finite products and coproducts;

[48]The popular psychologized reading of Hegel and what was meant by *Geist* may not have helped matters.

[49]Although he would no doubt disapprove, one can praise Lawvere's entire project as the triumph of algebra over ontology, in as much as what is meant by ontology is the branch of philosophy where one asserts by faith alone predicates which underlie all logical discourse, and acceptance of those predicates is due to the authority of a Very Important Philosopher (and the intellectual laziness of the adherent).

[50]Recall that this entails $f^* \dashv f_* : \mathcal{D} \to \mathcal{E}$, where f^* is left-exact.

(2) has a quality type L with $q : L \to B$ and $q_* = q_! : L \to B$;
(3) $q_* s_* = f_*$
then s_* is an **intensive quality on E**.

Remark. First, for some grounding, think of what quality type through which we have been factoring Π.

This is key, as a quality type is a full embedding of the base category on the left and right (as the adjoints are the same).[51] Furthermore, Lawvere notes that an extensive quality of an object $a : E$ has the same number of connected pieces as a. Finally, he rather cryptically suggests that the canonical extensive quality is **form**, but this can be taken literally to mean that form is simply a coproduct of connected components. Moreover, extensive quality types are found in any cohesive category by a theorem of Hurewicz (the statement and proof of which is given rather tersely in [11]).

In contrast to extensive quality, Lawvere rather helpfully notes that an intensive quality of an object $a : \mathcal{E}$ has the same number of points as a, before going on to provide a proof that any cohesive category has a canonical intensive quality. In the case of where one gets the canonical notion of **substance**, I suppose the rather intuitive idea that substance encompasses every particular instance of some object should suffice. Lawvere himself points out this distinction between "in the large" and "in the small" in traditional philosophical analysis. Again, thinking back to our topological example, the canonical *qualities* have thus far been discreteness and codiscreteness.

It is both remarkable and quotidian that he can precisely describe these concepts with category theory. Remarkable because the history of philosophy is rife with noble attempts at providing a sound formal ontology which allows one to reason about these concepts, and quotidian, because it really is not that surprising that these were sound ideas merely waiting for the appropriate mathematical ontology; remember, Mac Lane was well versed in the philosophical tradition of German Idealism.[52]

Yet, cohesion does not stop with topos theory.[53] Indeed, one can consider $(\infty, 1)-$categories with object classifiers (these would be $(\infty, 1)-$topoi, which [17] suggests are the correct objects to internalize homotopy type theory, subject to some open questions regarding coherence issues), and Schreiber and Shulman [17], proceed to provide the cohesion axioms in type theoretic manner via *reflective subfibrations* and $\sharp - \texttt{Types}$. Sadly, properly analyzing Schreiber and Shulman's work on cohesive homotopy type theory ([18], [17], [19],and [20], to cite a few) is beyond the scope (and time constraints) of this paper.[54] However, the authors of [17] more than ably speaks for themselves. The intrigued reader should now be equipped to read this paper, and many of the papers cited therein.

2.3. Some Motivating Examples

Example. The motivating example for this subject comes from Lawvere's analysis of Cantor's account of *Mengen* and *Kardinalen*, where a *Menge* can be thought of as the underlying structure on an ensemble of parts. Lawvere notes that *Menge* is often translated as *set*,[55] while Kardinale should be thought of as an abstract set.

The notion of **quantity** is an adjoint between discreteness and continuity given by $\flat \dashv \sharp$. These adjoint modalities capture the seeming paradox presented by Cantor in which elements of a set are distinct yet indistinguishable. In effect, this harkens back to the earlier example of $F \dashv U \dashv G : \texttt{Set} \to \texttt{Top}$, where $U \equiv \Gamma$ is the forgetful functor, and $F \equiv \texttt{Disc}$ and $G \equiv \texttt{coDisc}$.

Example. Similarly, one can capture the geometric notion of continuum geometry with the adjoint cylinder from $\int \dashv \flat$. This gives rise to the points to pieces transformation by composing the natural transformations $\eta \circ \varepsilon$, with η the unit of $\int$ and ε the co-unit of $\flat$. This is illustrated in Figure 1.

In particular, as $\int \equiv \texttt{Disc} \circ \Pi$, and $\Pi : \texttt{Top} \to \texttt{Set}$ takes connected components to their underlying sets, we can concretely see how $\int$ goes from "points" to "pieces".[56] For instance, if this transformation is an

[51] Hint: I've given the answer elsewhere.

[52] Indeed, the label of categories was inspired by Kant, who himself was drawing from the rich philosophical tradition of considering categories to be a maximally extensive class whose objects were affirmable predicates. Of course, if this description is maddeningly abstruse, there is always Type theory and diagrams, which we can verify as sensible!

[53] Otherwise, there wouldn't be this paper.

[54] As well as this author's skill as an expositor.

[55] While also lamenting that in acting as gatekeeper of sorts to Cantor, Zermelo may have while helped add to our confusion about this Mengen/Kardinalen paradox.

[56] Just ask yourself, "What is the image of $\texttt{Disc}(\Pi(X \coprod Y))$ in $\texttt{Top}$?", where X, Y are connected topological spaces.

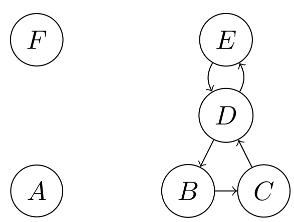
equivalence, ie $\flat \xrightarrow{\simeq} \int$, then $\mathbf{H}$ is infinitesimally cohesive, in the sense that objects are built from precisely one point in each cohesive piece.

2.4. Cohesion In Reflexive Graphs Recall from our earlier examples that `RGph` is a subcategory of the pre-sheaf category `PSh(A)`, where A is a category of two objects V, E and four maps, the two identities and $s, t : V \to E$. Moreover, `RGph` is a topos. Although a full verification of the adjoint triple of modalities will not be provided,[57] provided below is the adjoint quadruple:

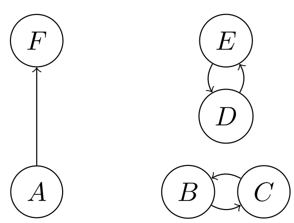
$$\Pi \dashv \texttt{Disc} \dashv \Gamma \dashv \texttt{coDisc}$$

which gives rise to cohesion in `RGph`, and a reasonably complete collection of proofs of subclaims to support that these functors define `RGph` as a cohesive topos. Going from right to left, we define the functors acting on objects as follows:

- `coDisc` : `Set` $\to$ `RGph` takes an underlying set a : `Set` to the presheaf $G_{a,c}$ such that $G_{a,c}(V) = a$ and G is a complete graph, i.e. for any two points $x, y : a$, there is are $e_1, e_2 : G_{a,c}(E)$ such that $s(e_1) = x = t(e_2)$ and $s(e_2) = y = t(e_1)$.
- Γ : `RGph` $\to$ `Set` takes the presheaf G to $G(V)$, the underlying set;
- `Disc` : `Set` $\to$ `RGph` takes a set a : `Set` to the presheaf $G_{a,d}$ such that $G_{a,d}(V) = a$ and $G_{a,d}(E)$ is a graph whose only edges are the self-directed edges of each vertex, i.e., a graph whose only edges are loops at a vertex.
- Π : `RGph` $\to$ `Set`, takes a pre-sheaf G to the quotient set $G(V)/\sim$ where $a \sim b$ if and only if a, b are vertices of a latent connected undirected subgraph $[g] \hookrightarrow G$. Informally, the latent undirected subgraphs are all the edges whose source and target are "forgotten", and an undirected subgraph is connected, if for each pair of vertices a, b in the subgraph, there is a path. For instance, Π takes the following graph to:



to the set $\{\{A\}, \{B, C, D, E\}, \{F\}\}$, while sending



to $\{\{A, F\}, \{B, C\}, \{D, E\}\}$. In both cases, this is merely sending the weakly connected path components to the set of their vertices. In particular, we recognize that for each pre-sheaf G, there are sub-sheafs, denoted $[g] \hookrightarrow G$, such that $[g]$ describes a weakly connected subgraph, and for $\bar{u} : G(V)/\sim$, $u : [g]$ for one, and only one $[g]$, which we will denote by $[g]_u$, so Π maps $[g]_u \mapsto \bar{u}$,

We claim (and rather informally prove),[58] that these form an adjoint quadruple $\Pi \dashv$ `Disc` $\dashv \Gamma \dashv$ `coDisc` and that Π as defined preserves finite products and that `Disc` and `coDisc` are full and faithful.

Let a : `Set` and G : `RGph`, going right to left as before:

- It suffices to notice the following:
  (1) If G is a reflexive graph and if H is any complete reflexive graph, than any function $f : G(V) \to H(V)$ induces a graph homomorphism as all edges of $G(E)$ are 'preserved' in a complete reflexive category, either mapping to new edges or loops at a vertex;

[57] Furthermore, Lawvere has already done so with some very terse, and very high level category theory elsewhere, such as in [11], in the case of undirected reflexive graphs.

[58] These sketches can be elaborated upon to properly establish the functoriality and adjointness of the pairs and the unit and co-unit natural transformations satisfying the zig-zag identities.

(2) We identify $\Gamma(G) = G(V)$;
(3) For every $a : \texttt{Set}$, $\texttt{coDisc}(a) = H_{a,c}$ and $H_{a,c}(a) = a$, and so $\texttt{Hom}_{\texttt{RGph}}(G, \texttt{coDisc}(a)) = \texttt{Hom}_{\texttt{RGph}}(G, H_{a,c})$.

From here, it is a matter of applying (1) to the hom-sets $\texttt{Hom}_{\texttt{Set}}(\Gamma(G), a) \cong \texttt{Hom}_{\texttt{RGph}}(G, H_{a,c})$.

- Similarly, it suffices to notice the following:
  (1) If G is a reflexive graph and if H is any reflexive graph consisting solely of the loops on its vertices, than any set-function $f : H(V) \to G(V)$ induces a graph homomorphism as all loops of $H(E)$ are 'preserved' by the mapping of the set of vertices;
  (2) We identify $\Gamma(G) = G(V)$;
  (3) For every $a : \texttt{Set}$, $\texttt{Disc}(a) = H_{a,d}$ and $H_{a,d}(a) = a$, and so $\texttt{Hom}_{\texttt{RGph}}(\texttt{Disc}(a), G) = \texttt{Hom}_{\texttt{RGph}}(H_{a,d}, G)$.

  From here, it is a matter of applying (1) to the hom-sets $\texttt{Hom}_{\texttt{Set}}(a, \Gamma(G)) \cong \texttt{Hom}_{\texttt{RGph}}(H_{a,d}, G)$.
- Next, it suffices to notice the following:
  (1) $\texttt{Disc}(a)(V) = a$.
  (2) Let $f : G \to H$ with $f = (f_1, f_2)$, then $\Pi(f) : G(V)/\sim \to H(V)/\sim$ is defined by the mapping $f \mapsto \bar{f}_1$, where $\bar{f}_1$ is the induced mapping of f_1 such that

$$\begin{array}{ccc} G(V) & \xrightarrow{f_1} & H(V) \\ {\scriptstyle q_G}\downarrow & & \downarrow{\scriptstyle q_H} \\ G(V)/\sim & \xrightarrow[\bar{f}_1]{} & H(V)/\sim \end{array}$$

  commutes, where q_G, q_H are the respective canonical quotient maps.
  (3) Given any graph homomorphism $f : G \to \texttt{Disc(a)}$, since graph homomorphisms preserve edges, if $u, v : G(V)$ have an edge $e : G(E)$, and $f_1(u) = x$ for some point $x : a$, then necessarily $f_1(v) = x$ and $f_2(e)$ is the self directed edge at x. If $f_1(v) \neq x$, then there is no possible edge for e to be mapped to in the discrete graph, and thus f would not be a graph homomorphism. This is to say that f_2 is fully determined by the map f_1, and hence the graph homomorphism f is determined by the set homomorphism f_1.
  (4) Let $a : \texttt{Set}$. Given $f = (f_1, f_2) : G \to \texttt{Disc(a)}$ is a graph homomorphism, we see that f_1 uniquely determines a set-function $f_1^\sim : G(V)/\sim \to a$. Since f is a graph homomorphism, f_2 must take all edges of a weakly connected directed subgraph $[g] \hookrightarrow G$ to the same self-directed edge in $\texttt{Disc}(a)$ and all vertices of $[g](V)$ to the associated vertex of that edge. So, for $\bar{u} : G(V)/\sim$, where $u : [g]$ for some unique $[g] \hookrightarrow G$, we define $f_1^\sim$ by

$$\bar{u} \mapsto s(f_2(q^{-1}(\bar{u})))$$

  To see why this is unique, if f_1, g_1 are graph homomorphisms that determine the same $h^\sim$ function, then they must also agree on their mapping of each $u : G(V)$ to $\texttt{Disc}(V)(a)$, which means as set functions $f_1(u) = g_1(u)$ for all u, and hence $f_1 = g_1$.
  (5) Moreover, if we were given $f_1^\sim : G(V)/\sim \to a$, we can uniquely induce a set function $f_1 : G(V) \to \texttt{Disc}(a)(V)$ as follows: given that every vertex in $u : G$ belongs to one and only one weakly connected subgraph $[g]$, denoted $[g]_u$, and for any weakly connected subgraph $[g]$ of G, there is by definition a vertex $x : a$ such that $f_1^\sim(q([g])) = x$, for all $u : G(V)$, define f_1 by $u \mapsto f_1^\sim(q([g]_u))$. Since $\texttt{Disc}(V) = a$, we find this defines $f_1 : G(V) \to \texttt{Disc}(a)(V)$.
  Furthermore, by (2), this $f_1^\sim$ determines the graph homomorphism f.
  (6) This gives us for each $a : \texttt{Set}$ and $G : \texttt{RGph}$, $\alpha : \texttt{Hom}_{\texttt{RGph}}(G, \texttt{Disc}(a)) \to \texttt{Hom}_{\texttt{Set}}(\Pi(G), a)$ by $\alpha(f) = \alpha((f_1, f_2)) = f_1^\sim$.
  (7) If $\alpha(f) = \alpha(g)$, then $f_1^\sim = g_1^\sim$. But this means that f_1 and g_1 map each individual weakly connected components to the same vertex in a, from which we can see $f_1 = g_1$ as in each weakly connected component in G every vertex is mapped to the same vertex in a. Hence α is injective.
  (8) Similarly, for each $\varphi : \texttt{Hom}_{\texttt{Set}}(\Pi(G), a)$, we find there is an f such that $\alpha(f) = f_1^\sim = \varphi$. We do this as follows: set $f_1^\sim := \varphi([g])$ for each weakly connected subgraph $[g]$. By (4), we see this determines a graph homomorphism f.

  By (7) and (8) $\alpha : \texttt{Hom}_{\texttt{RGph}}(G, \texttt{Disc}(a)) \cong \texttt{Hom}_{\texttt{Set}}(\Pi(G), a)$.

- Finally, it bears remarking that if $u : [g]$ then we can identify $\bar{u}$ with $[g]$.

We have thus established that these do indeed define an adjoint quadruple

$$\Pi \dashv \texttt{Disc} \dashv \Gamma \dashv \texttt{coDisc}$$

Now to see "cohesion", we must prove the following claims:

Claim 6. *$\texttt{Disc}$ is full and faithful.*

Proof. To check that $\texttt{coDisc}$ is full and faithful, it suffices to notice that

$$\texttt{Hom}_{\texttt{Set}}(a, b) \cong \texttt{Hom}_{\texttt{RGph}}(\texttt{Disc}(a), \texttt{Disc}(b))$$

follows immediately from the following fact about maps between discrete graphs: any set morphism between the sets of vertices of two discrete graphs extends to a graph homomorphism between discrete graphs. The uniqueness of the graph homomorphisms follows immediately from the uniqueness of the set morphisms. □

Claim 7. *$\texttt{coDisc}$ is full and faithful.*

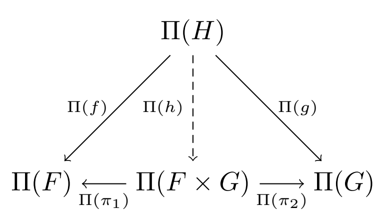
Proof. To check that $\texttt{coDisc}$ is full and faithful, it suffices to notice that

$$\texttt{Hom}_{\texttt{Set}}(a, b) \cong \texttt{Hom}_{\texttt{RGph}}(\texttt{coDisc}(a), \texttt{coDisc}(b))$$

follows immediately from the following fact[59] about maps between complete graphs: any set morphism between the sets of vertices of two complete graphs extends to a graph homomorphism. The uniqueness of the graph homomorphisms follows immediately from the uniqueness of the set morphisms. Moreover, because every vertex is connected and the graph is reflexive, any set function assignment can be permitted without worrying about "missing" and edge. □

Claim 8. Π *preserves finite products.*

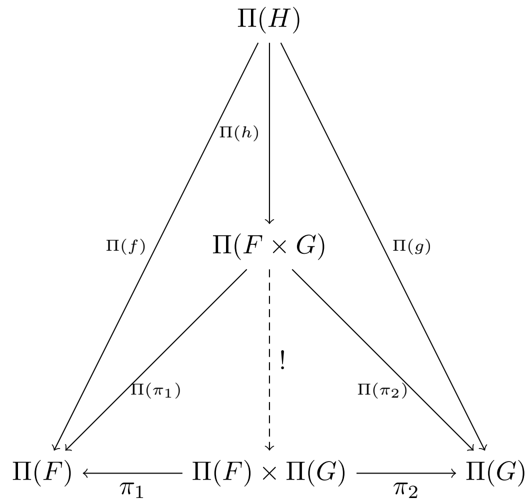
Proof. (Informal) As has now been shown, both $\texttt{Set}$ and $\texttt{RGph}$ as toposes, have finite products. We thus wish to show that given a product in $\texttt{RGph}$, under $\Pi(-)$ we have that



commutes.

[59]Mentioned, and slightly sketched earlier.

First observe that in **Set**,



From here, we wish to show that $! : \Pi(F) \times \Pi(G) \to \Pi(F \times G)$. In fact, this is tantamount to showing there is a unique map from $(F(V)/\sim \times G(V)/\sim) \to (F \times G)/\sim$.

To go about this, first the reader is encouraged to verify that the product of two weakly connected subgraphs $[f], [g]$ in G and H respectively is a weakly connected subgraph in $G \times H$, and moreover that every weakly connected subgraph in $G \times H$ is determined by the product of two weakly connected subgraphs.[60] Then, define $\varphi : \Pi(F) \times \Pi(G) \to \Pi(F \times G)$ by $(\bar{u}, \bar{v}) \mapsto q(([f]_u \times [g]_v)(V))$. Now, without loss of generality, considering that $\Pi(\pi_1) : (F \times G)(V)/\sim \to F(V)/\sim$, we find that $\Pi(\pi_1)(\bar{u}, \bar{v}) = \bar{u}$. Similarly, we find that $\Pi(\pi_2)(\bar{u}, \bar{v}) = \bar{v}$. Moreover, as φ maps the product of equivalence classes to the quotient of the set of vertices of the product of the respective connected subgraphs, it is reasonable to expect that this is the unique map doing so. Indeed, the interested reader can confirm this.

□

Hopefully, the reader better understands how **RGph** is a cohesive topos. For a concrete illustration, using an earlier reflexive graph, we can see "cohesion" through the appropriate modalities as follows:

[60]Consider that an edge e_{FG} is composed of e_F and e_G, and if $[f \times g]$ is a weakly connected subgraph of $F \times G$, then there is a path between all vertices in the underlying undirected graph. If e_F and e_G are edges in weakly connected subgraphs, there are corresponding e'_F, e'_G which form a path with e_F and e_G (these may simply be the self directed edge). Without loss of generality, suppose $t(e_F) = s(e'_F)$ and $t(e_G) = s(e'_G)$, then the product $e_F \times e_G = e_{FG} = \{\{s(e_F), s(e_G)\}, \{t(e_F), t(e_G)\}\}$ 'connects' with $e'_F \times e'_G = e'_{FG}$.

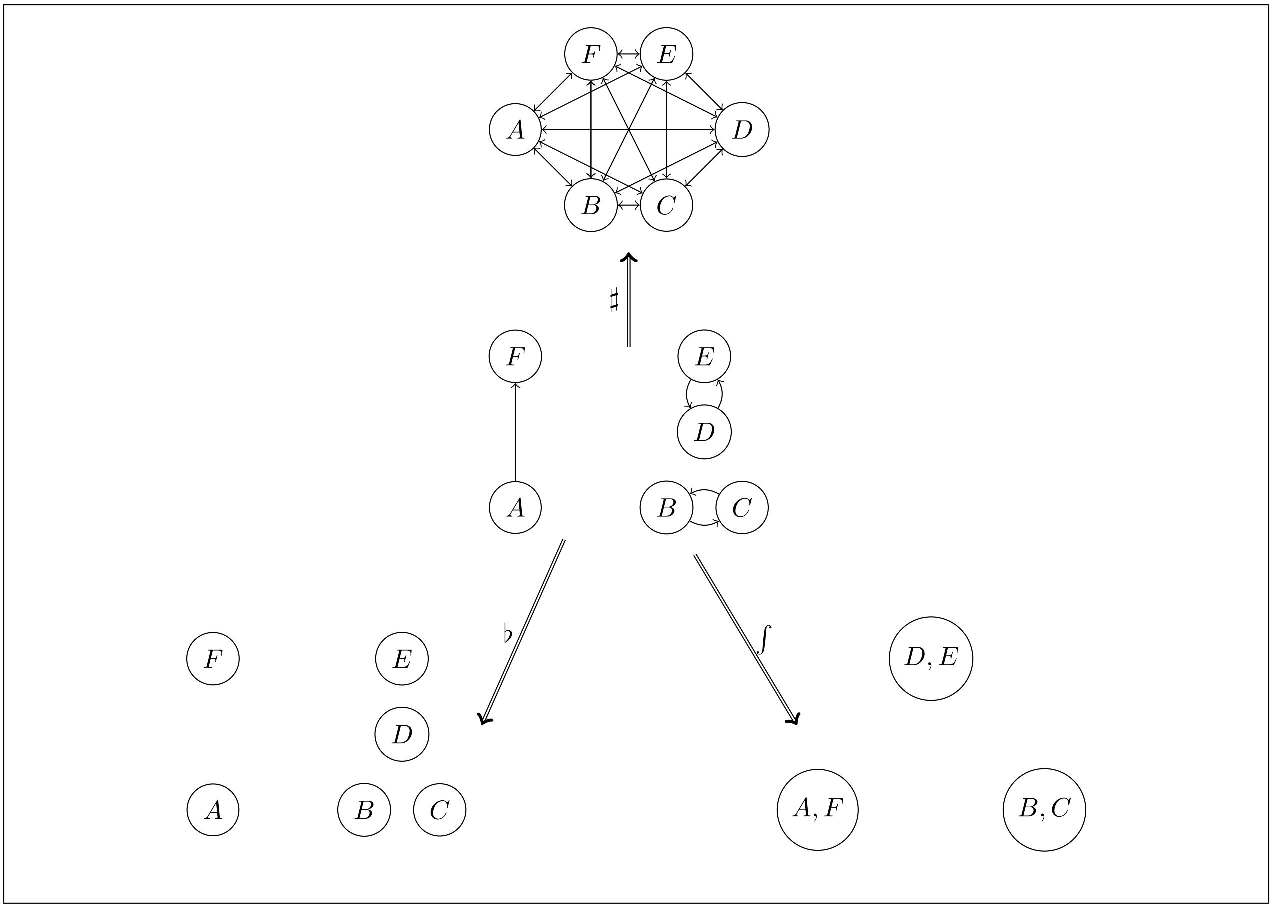
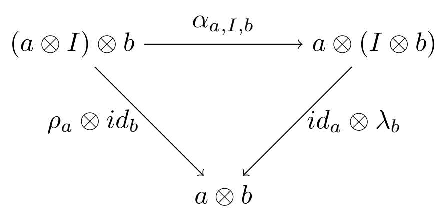


Figure 1: In this case we take an earlier graph and show how it changes under the $\sharp$, $\flat$, and $\int$ modalities. One can quite literally now see how $\flat \to \int$ is a matter of "points" to pieces" under the composition of $\eta \circ \varepsilon$ where η is the unit of $\int$ and ε is the co-unit so that in this case we have the component composition $\varepsilon_G \to 1_G \to \eta_G$. Hopefully, this helps vividly illustrate how discrete objects are both reflective and coreflective, with $\flat$ the discrete coreflector and $\int$ the discrete reflector. Finally, this is where Lawvere suggests one can find a formalization of the "unity of opposites".

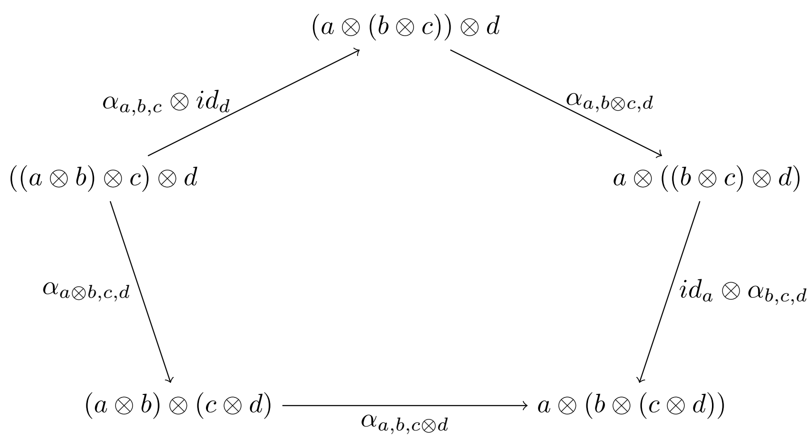
3. Endnote: An Aside On Monoidal Categories

however, it is worth studying the following definition and examples in their own right if one wants to appreciate *enriched categories*:

Definition. A **monoidal category** is a category A equipped with the 5−tuple $(\otimes, I, \lambda, \rho, \alpha)$ where:
(1) $\otimes : A \times A \to A$, called the **monoidal product** (or **tensor product**) is a bifunctor;
(2) $I : A$ called the **unit object** or **tensor unit**;
(3) A natural transformation λ called the **left-unitor**, with components $\lambda_a : I \otimes a \to a$;
(4) A natural transformation ρ called the **right-unitor**, with components $\rho_a : a \otimes I \to a$;
(5) A natural transformation α called the **associator** with component $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$.
which satisfy the following coherence conditions:



and



Example. The most immediately sensible example of a monoidal category is the category $R-$`mod` over a commutative ring R, where $\otimes_R$, the tensor product of modules, is the monoidal product and the ring R is the unit (as R is a module over itself). Immediately it follows that the category $\texttt{Vec}_k$ of vector spaces over a field k and `Ab`, the category of abelian groups, with $\mathbb{Z}$ as the unit, are two other examples of monoidal categories.

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