NOTES ON CATEGORY THEORY: FROM THE BASICS TO DERIVED FUNCTORS

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Abstract. These notes formed the basis of a two-part lecture titled *Category Theory: Or How I Learned To Stop Worrying and Love the Diagram* that I delivered at the University of Waterloo for the Group Cohomology Learning Seminar, in early October, 2013. They reflect my own work, and are meant as an introduction to Category Theory, with an aim of motivating derived functors in the study of group cohomology. Familiarity with abstract algebra, and group theory in particular, is assumed.

1. *What Is A Category?*

Definition. A *Category* $\mathcal{C}$ consists of two classes, *Objects*, denoted $\text{Ob}(\mathcal{C})$, and *Arrows*, denoted $\text{Ar}(\mathcal{C})$, which satisfy the following:

(i) For each object $X \in \text{Ob}(\mathcal{C})$, there is an arrow $1_X : X \to X$ called the *identity arrow*.
(ii) For each arrow f, we have the following objects: $dom(f)$, the *domain* of f, and $cod(f)$, the *co-domain* of f. Setting $A = dom(f)$ and $B = cod(f)$, we view $f : A \to B$ to indicate an arrow's source and target (domain and codomain respectively).
(iii) Given arrows $f, g \in Ar(\mathcal{C})$ such that $cod(f) = dom(g)$, there is an arrow $g \circ f$ with $dom(g \circ f) = dom(f)$ and $cod(g \circ f) = cod(g)$, called the *composite* of f and g.
(iv) (Associativity) For arrows $f, g, h \in \text{Ar}(\mathcal{C})$
$$h \circ (g \circ f) = (h \circ g) \circ f$$
(v) (Unit)With $X = dom(f)$ and $Y = cod(f)$
$$f \circ 1_X = f = 1_Y \circ f$$

Notation. For any two objects $X, Y \in \text{Ob}(\mathcal{C})$, we denote the class of arrows between X and Y by $\mathcal{C}(X, Y)$.

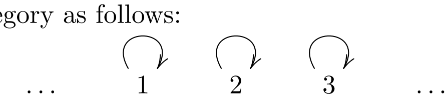
Notation. Throughout these notes, unique morphisms are indicated by !. Occasionally, they will be further demarcated by subscripts.

Remark. I cannot emphasize enough the importance of this arrow-centric perspective. In category theory, objects are of secondary importance to the arrows between them. Indeed, (i) suggests that we can do away with looking at objects altogether, as a requirement for an object to be in a given category is that an identity arrow exists in the category.

Remark. We can identify certain categories whose classes of objects and arrows are sets are called *small categories* . In the case where we are working in small categories, it is customary to refer to our set of arrows by $\text{Hom}(\mathcal{C})$. Moreover, a category is *locally small* if for arbitrary objects $X, Y \in \text{Ob}(\mathcal{C})$, the homomorphism class $\text{Hom}_{\mathcal{C}}(X, Y)$ is a set[1]. We will chiefly be working with locally small categories throughout these notes. When either $\text{Ob}(\mathcal{C})$ or $\text{Ar}(\mathcal{C})$ are proper classes, we are working with a *large* category.

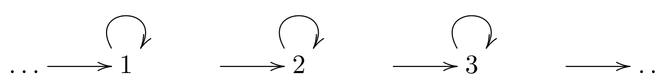
Some examples of category

Example. The simplest categories that we can work with, and one that also has an explicit model theoretic structure, is a trivial category realizing the empty language $\mathcal{L} = \{\}$. Any structure which realizes $\mathcal{L}$ is one whose atomic formula consist entirely of $t_1 = t_2$. As a locally small category, our set of objects would be some underlying universe M, say $\mathbb{Z}$, and our set of arrows would consist solely of the identity arrows on the objects. We can visualize this category as follows:

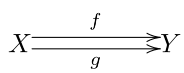


[1]It is not only possible, but quite common that we are working in a locally small category which is not small itself. Just consider the category of sets!

Example. Building off the previous example, one of the most fundamental examples we have of a category is one with a *poset* structure.
This category is determined by the requirement that every hom-set has at most one element. This category in effect consists entirely of objects and a partial order relation denoted $\leq$, which we use in lieu of individual morphisms. To see why a partial order relation $\leq$ can be used in lieu of explicit morphisms between objects, we recall that partial order relations must be reflexive, antisymmetric, and transitive.
Since $\leq$ is reflexive, $\leq$ satisfies our requirement that all objects have an identity morphism. Moreover, by transitivity, $\leq$ satisfies the requirement that the composite morphism exists. To see this, consider $A \leq_{A,B} B$ and $B \leq_{B,C} C$, where $\leq_{A,B} \in \text{Hom}_{\mathcal{C}}(A, B)$ and $\leq_{B,C} \in \text{Hom}_{\mathcal{C}}(B, C)$. Then $\leq_{B,C} \circ \leq_{A,B} \in \text{Hom}_{\mathcal{C}}(A, C)$. By our requirement that every hom-set has at most one morphism, $\leq_{B,C} \circ \leq_{A,B} = \leq_{A,C}$.
Since this subscripting can become quite tedious, it is obvious why we can substitute each respective morphism with our partial order relation $\leq$.
Moreover, we see that any any poset category is necessarily a structure which realizes the language $\mathcal{L}_{ord} = \{\leq\}$. Some explicit examples are the ordinals. Another would be the example from above, only now:



Example. One of the main motivating examples in category theory is the category of sets. The category of sets, denoted **Set**, has sets for objects and has functions between sets as its morphisms. This category is *locally small*, but it is not a small category for obvious reasons.[2]. It should be noted that for sets X, Y we can have $f, g \in \text{Hom}_{\mathbf{Set}}(X, Y)$. We visualize this as the following simple diagram:

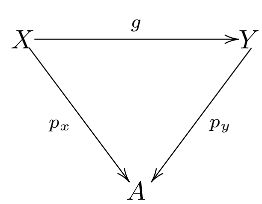


If $f = g$, we say the above diagram *commutes*.

Example. The category of groups, denoted **Grp** has groups for objects and group homomorphisms for its arrows.

Example. The category of abelian groups, denoted **Ab** has abelian groups for objects and group homomorphisms for its arrows.

Example. Although we cannot define *comma category* in full generality at this point, one useful example in particular is the *slice category* $\mathcal{C} \downarrow A$, defined for some $A \in \text{Ob}(\mathcal{C})$. For a locally small category $\mathcal{C}$, the slice category is the category whose objects consist of arrows $f \in \text{Hom}_{\mathcal{C}}(X, A)$ and whose arrows, are arrows $g : X \to Y$ such that for objects $p_x : X \to A$ and $p_y : Y \to X$ in $\mathcal{C} \downarrow A$, the following diagram



commutes.

2. *Universal Properties*

One of the more powerful aspects of category theory is that it enables us to both define and study *universal properties*, via diagrams. In particular, many useful definitions in category theory are provided by their universal mapping properties. Here are some of the most fundamental and useful properties, beginning first with some terminology regarding arrows.

[2]Russell's paradox. Although to be fair, this reason was not obvious for the first few decades of set theory.

2.1. Morphisms

Definition. A *monic* arrow, or *monomorphism*, is a left-cancellative arrow $f : X \to Y$ such that for all arrows $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

We identify monic arrows with

$$Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

Within the category **Set**, monic arrows are precisely injections. Similarly, we identify

Definition. A *epic* arrow, or epimorphism, is a right-cancellative arrow $f : X \to Y$ such that for all arrows $g_1, g_2 : Y \to Z$,

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

We identify epic arrows with the following diagram

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

Similarly, in **Set** we identify epic arrows with surjections.

Remark. One of the more remarkable results that one comes across when first learning category theory is that the axiom of choice is equivalent to the following: all epimorphisms *split.* That is, a category satisfies the axiom of choice if for any epimorphism $e : X \to Y$ in $\mathcal{C}$, there is an arrow $s : Y \to X$ called a *section* satisfying $e \circ s = 1_Y$. Exploring this in greater detail is outside of the scope of these notes, but the reader is encouraged to see why this is the case. To see why, consider epimorphisms in the **Set**, where we know set

Definition. An arrow $f : X \to Y$ in category $\mathcal{C}$ is an *isomorphism* if there is an arrow $g : Y \to X$ such that

$$g \circ f = 1_X \text{ and } f \circ g = 1_Y$$

We say that X is isomorphic to Y and denote this by $X \cong Y$, if an isomorphism exists between them

Theorem 1. *Inverses are unique.*

Proof. Suppose that $f : X \to Y$ is an isomorphism. We note an inverse arrow of f by $g = f^{-1}$. Now let $g, g' : Y \to X$ be two inverse morphisms for f.

$$\begin{aligned} g &= g \circ 1_Y \\ &= g \circ (f \circ g') \\ &= (g \circ f) \circ g' \\ &= 1_X \circ g' \\ &= g' \end{aligned}$$

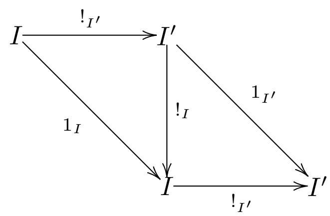
Hence $g = g'$. □

The reader should take care to note that isomorphisms are defined entirely in an abstract, category theoretic fashion, and not, as is most often the case in ones mathematical experience, as a bijection between objects. To fully see the utility of this perspective, consider the following example:

Example. The category of posets, denoted **Pos**, has posets for its objects and monotonic maps for its arrows. Now consider $\mathbb{Z}$ as an object in **Pos** with the standard ordering. Then $f : \mathbb{Z} \to \mathbb{Z}$ mapping $x \mapsto x+1$ is an order preserving bijection. However it is NOT an isomorphism.

2.2. Initial and Terminal

Definition. An *initial object* in a category $\mathcal{C}$ is some object $I \in \text{Ob}(\mathcal{C})$ such that for any object $X \in \text{Ob}(\mathcal{C})$, there is a unique morphism $!_X : I \to X$. Moreover, where they exist, initial objects are unique up to isomorphism. We can verify this by checking that the following diagram commutes for initial objects I, I'



Because each object has an identity arrow, the unique arrow from an initial object to itself is the identity arrow. Moreover, composition of arrows with an initial object as a source is unique by this defining property, hence we find that $!_I \circ !_{I'} = 1_I$. Similarly, we verify that the rest of the diagram commutes and find that initial objects $I \cong I'$.

Definition. A category $\mathcal{C}$ has a *terminal object* T, if for every $X \in \text{Ob}(\mathcal{C})$, there is a unique arrow $!_X : X \to T$. We can verify that terminal objects are isomorphic to one another following as above.

Remark. Terminal objects and epimorphisms are both the earliest examples of *duals*, respectively of initial objects and monomorphisms. We shall make this notion explicit in a little bit.

Definition. A *zero object* is an object $0 \in \mathcal{C}$ that is both initial and terminal.

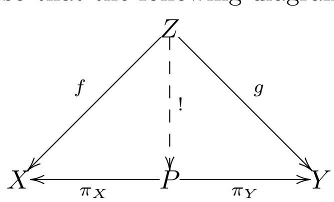
Example. A crucial zero object, is the one element group in the category **Grp**, and similarly in $\mathbf{Vec}_{\mathbb{K}}$, the category of vector spaces over the field $\mathbb{K}$. The reader can quickly verify that in the former case, there is a unique homomorphism from the identity element to any group and for any group there is a unique homomorphism to the identity element.

2.3. Other Important Constructions

Definition. In a category $\mathcal{C}$, a *product* for objects $X, Y \in \text{Ob}(\mathcal{C})$ consists of an object P, with a pair of maps π_X, π_Y from $\pi_X : P \to X$ and $\pi_Y : P \to Y$ satisfying the following universal mapping property:
Given any diagram of the form:

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

There exists a unique arrow $! : X \to P$ so that the following diagram

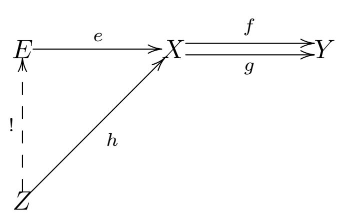


commutes. Consequently, $\pi_X \circ ! = f$ and $\pi_Y \circ ! = g$.

Example. The category theoretic notion of product is just the notion of a cartesian product when considering **Set**.

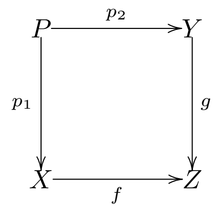
Remark. So far we have placed in the background the fact that for universal mapping properties that there are two parts: *existence* and *uniqueness*.

Definition. A category $\mathcal{C}$ has an equalizer (E, e), where e is an arrow $e : E \to X$ such that

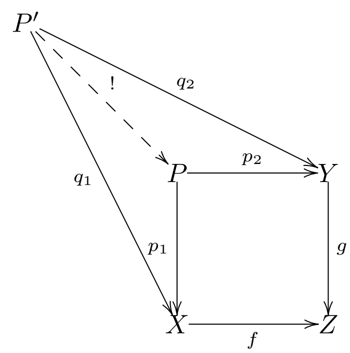


commutes. In more familiar terms, (E, e) is a pair of object and arrow such that $f \circ e = g \circ e$ for arrows $f, g : X \to Y$, so that for any object Z and arrow $h : Z \to X$, if $f \circ h = g \circ h$, then there exists unique $! : Z \to E$ such that $e \circ ! = h$

Definition. A *pullback* (or *fibre product*) of morphisms f, g, where $cod(f) = cod(g)$, consists of an object P and morphisms $p_1 : P \to dom(f)$ and $p_2 : P \to dom(g)$ such that the following square, where $dom(f) = X$, $dom(g) = Y$ and $cod(f) = Z$.



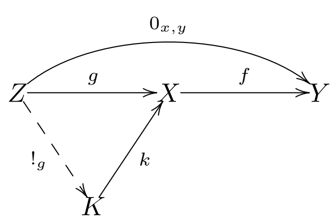
commutes, and the triple (P, p_1, p_2) satisfies the following universal property:
If the triple (P', q_1, q_2) forms a commutative diagram with our pair (f, g), then there exists a unique $! : P' \to P$ such that $p_1 \circ ! = q_1$ and $p_2 \circ ! = q_2$, such that the following diagram



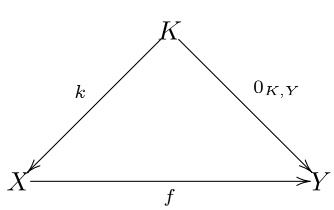
commutes.
The pullback is often denoted by $P = X \times_Z Y$.

Definition. A category $\mathcal{C}$ has a *zero morphism*, denoted by $0_{X,Y} : X \to Y$, if $\mathcal{C}$ has a zero object and $0_{X,Y} : X \to Y$, and $0_{X,Y}$ factors uniquely through the zero object.

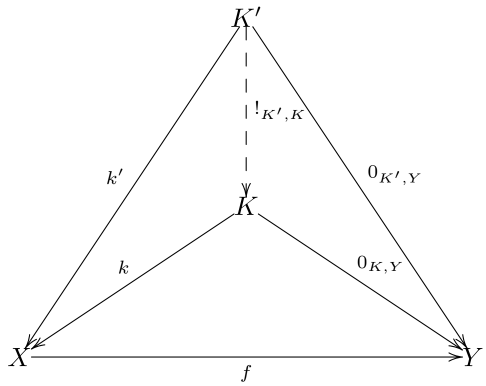
Definition. Let $\mathcal{C}$ be a locally category with a zero object, finite products, and where $\mathrm{Hom}_{\mathcal{C}}(X, y)$ is endowed with an abelian group structure so that the composition of maps is bilinear. Let $f : X \to Y$ be a morphism. Then a morphism $k : K \to A$ is a *kernel* of f if $f \circ k = 0$ and for all $g : Z \to X$ such that $f \circ g = 0$, there exists a unique $!_g : Z \to K$ so that



commutes. We can characterize a kernel k of f by the following universal property: $f \circ k = 0_{K,Y}$ so that

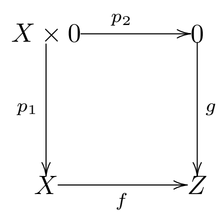


commutes, and given any $k' : K' \to X$ such that $f \circ k' = 0_{K',Y}$, there is a unique $!_{K',K} : K' \to K$ such that



commutes.

Example. In a category $\mathcal{C}$ with zero objects (say the **Grp**), we can combine these definitions to describe the fibred product $X \times_Y 0$ with maps $f : X \to Y$ and $0_{0,Y}$ as the kernel of f. Notice that:



As this diagram commutes, $f \circ p_1 = 0_{X \times 0,B}$, the zero map. The reader can then verify from an elementary perspective in **Grp** that the diagram commutes only when $X \times 0$ consists of the elements of X that f sends to 0.

3. Duals and Opposites

Within the language of category theory, there is a notion of duality: for any statement ϕ, if ϕ follows from the axioms of category theory, then ϕ^*, the *dual* of ϕ, also follows. That is: $T_{cat} \models \phi \Rightarrow T_{cat} \models \phi^*$, where T is a theory of the axioms of category theory. But what does this really mean diagrammatically? It's actually quite simple. For a statement ϕ rendered diagrammatically, say

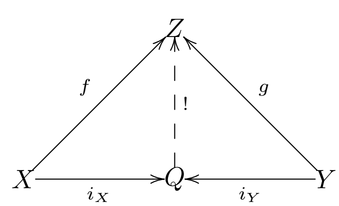
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

The dual of this mapping is:

$$X \xleftarrow{f} Y \xleftarrow{g} Z$$

The dual of a construction is obtained as a diagram simply by reversing the direction and the order of composition of arrows. The reader can go back to verify that the universal mapping properties for epimorphisms and terminal objects satisfy this notion of duality with their respective duals. Here are some other dual notions to constructions described above.

Definition. A category has *co-product* Q of objects X, Y if there are a pair of morphism (i_X, i_Y) so that $X \overset{i_X}{\to} Q \overset{i_Y}{\leftarrow} Y$ and if for any Z, and pair of arrows $f : X \to Z$ and $g : Y \to Z$, there is a unique arrow $! : Q \to Z$



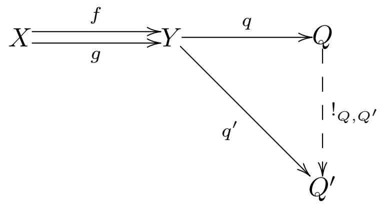
The co-product is denoted by the diagram $X \xrightarrow{i_X} X + Y \xleftarrow{i_Y} Y$; the arrows i_X, i_Y are called the *injections*.

Example. While in **Set**, products are the natural cartesian products, co-products are disjoint unions of sets. Consider

$$X + Y = \{(x, 1) : x \in X\} \cup \{(y, 2) : y \in Y\}$$

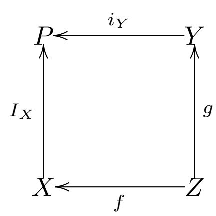
with the injective maps $i_X(x) = (x, 1)$ and $i_Y(y) = (y, 2)$.

Definition. Given a pair of arrows $f, g : X \to Y$, a *co-equalizer* can be defined as the pair (Q, q), where $q : Y \to Q$, such that $q \circ f = q \circ g$, which satisfy the following universal property:

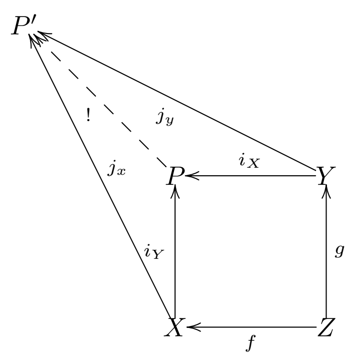


commutes.

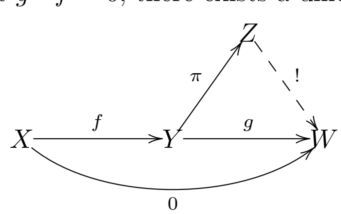
Definition. A category $\mathcal{C}$ has pushouts (or a cofibred coproduct) for two arrows f, g so that $dom(f) = dom(g) =: Z$ if there is a triple (P, i_X, i_Y) such that



commutes. Pushouts satisfy the following universal mapping property:



Definition. In a locally small category with zero objects, finite co-products and where each $\mathrm{Hom}_{\mathcal{C}}$ has an abelian group structure, for any morphism $f : X \to Y$, a morphism $\pi : Y \to Z$ is a *co-kernel* of f if $\pi \circ f = 0$ for all morphisms $g : Y \to W$ such that $g \circ f = 0$, there exists a unique $! : Z \to W$ so that



commutes

Remark. It should be noted that although in common undergraduate and first year graduate courses that the kernel is thought of as some subset of a domain of a function, or (in category theoretic terms) as a subobject of an object, in category theory we consider the kernel and cokernel in terms of morphisms. This seeming discrepancy is reconciled by the universal properties defining each so that we can talk about the kernel and the cokernel of a morphism.

Although we have so far looked at dual constructions to already familiar constructions, we can take this even further and look at the dual of a category.

Definition. For a category $\mathcal{C}$, the *dual* or *opposite category*, denoted $\mathcal{C}^{op}$ consists of two classes, the objects of $\mathcal{C}$ and an arrow class whose arrows are those of $\mathcal{C}$, but with their directions reversed. To illustrate this, for $f \in \mathcal{C}(X,Y)$, the corresponding arrow is $f^{op} : Y \to X$. We recognize this as $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$

4. Functors

The real power of category theory is not studying properties of categories in themselves (which is rather dull when you think about it– it would better serve a mathematician to study a category of interest by referring to the respective theory behind it), but in the maps between categories, and the degree that these maps preserve the structure of the source category.

Definition. For two categories $\mathcal{C}, \mathcal{D}$, a *covariant functor* is a mapping $F : \mathcal{C} \to \mathcal{D}$ that assigns each $X \in \text{ Ob }(\mathcal{C})$ to an object $F(X) \in \text{ Ob }(\mathcal{D})$ and for every pair of objects X, Y in $\mathcal{C}$ a function:

$$\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$$

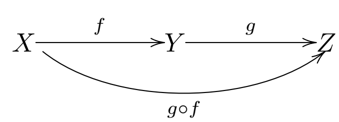
Moreover, F preserves identities and composition. That is $F(1_X) = 1_{F(X)}$ for all $X \in \text{ Ob }(\mathcal{C})$ and

$$F(g \circ f) = F(g) \circ F(f)$$

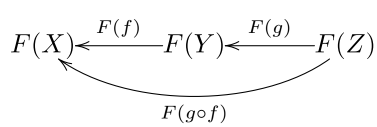
for all $X, Y, Z \in \text{ Ob }(\mathcal{C})$ and for all $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$.
Similarly, we define a *contravariant functor* F as above with the exception that we reverse composition. That is,for all $X, Y, Z \in \text{ Ob }(\mathcal{C})$ and for all $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$:

$$F(g \circ f) = F(f) \circ F(g)$$

Diagrammatically, a contravariant functor maps a composition of arrows $g \circ f$ in $\mathcal{C}$

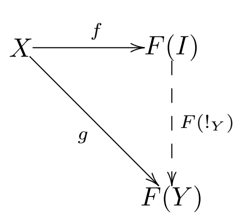


is mapped to



in $\mathcal{D}$.

Definition. Now we can define a universal property in greater generality. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor. Let $X \in \text{ Ob }(\mathcal{D})$. An *initial morphism* from X to F is an initial object (I, f) in the comma category $(X \downarrow F)$ of morphisms from X to F where I is an object of $\mathcal{C}$ and $f : X \to F(I)$ is an arrow in $\mathcal{D}$ satisfying the *initial property.*
The *initial property* is as follows: if $Y \in \text{ Ob }(\mathcal{C})$ and $g : X \to F(Y)$ is an arrow in $\mathcal{D}$, then there exists a unique $!_Y : I \to Y$ such that



commutes. We similarly define a *terminal morphism* and the *terminal property* as the dual construction. In the literature, *universal morphisms* refer an initial or terminal morphism, and *universal properties* respectively refer to initial or terminal properties.

Definition. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between locally small categories is *faithful* if for all objects $X, Y \in \text{Ob}(\mathcal{C})$, the induced function

$$\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

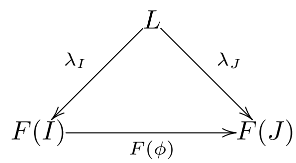
is an injection. If the induced function is surjective for all X, Y, then the functor F is *full*. This is to say that full and faithful functors F are bijective on morphism and thus preserve isomorphism classes.

Definition.

Many of the universal properties described above are particular instances of a categorical *limit* (and in the dual case *co-limit*). Explicitly, we define limits and co-limits as follows:

Definition. Let $F : \mathfrak{I} \to \mathcal{C}$ be a covariant functor from some index category $\mathfrak{I}$. An object $L \in \text{Ob}(\mathcal{C})$ is the *limit* of F if there are morphisms $\lambda_I : L \to F(I)$ for all $I \in \text{ob}\mathfrak{I}$ which satisfy

(i) If $\phi : I \to J$ is a morphism in $\mathfrak{I}$, then $\lambda_J = F(\phi) \circ \lambda_I$ so that



(ii) L is terminal with respect to this property, so that for any other object X with morphisms μ_I satisfying (i), then there is a unique morphism $X \to L$ so that all diagrams commute.

Definition. We define *colimits* of a functor $F : \mathfrak{I} \to \mathcal{C}$ as the dual of the limit. We identify the colimit with the direct limit of F, which we denote $\varinjlim F$. The colimit is an object $L \in \text{Ob}(\mathcal{C})$ such that there are morphisms $\lambda_I : F(\mathfrak{I}) \to \mathcal{C}$ for all $I \in \text{ob}(\mathfrak{I})$ such that $\lambda_I = \lambda_J \circ F(\phi)$ for all $\phi : I \to J$, and L is initial with respect to this property.

What are some examples of functors?

Example. One of the simplest functors is the *forgetful functor* $F : \textbf{Grp} \to \textbf{Set}$. This F sends groups G to their underlying sets, and group homomorphisms f to set functions $F(f)$, which in this case, are simply the group homomorphisms themselves.
In general, forgetful functors "lose" some structure of their source category. Forgetful functors can be thought of as maps between languages. Indeed, there are different kinds of forgetful functors. $F : \textbf{Ring} \to \textbf{Set}$ and $G : \textbf{Ring} \to \textbf{Ab}$ are both forgetful functors from the category of rings to the category of sets and abelian groups respectively.

Example. In the category of R-modules, denoted **R-mod**, for every R-module N, the tensor product $\otimes_R$ defines a covariant functor $- \otimes_R N : \textbf{R} - \textbf{mod} \to \textbf{R} - \textbf{mod}$ sending objects

$$M \mapsto M \otimes_R N$$

and which induces on R-module homomorphisms $f : M_1 \to M_2$ an R-linear map

$$f \otimes N : M_1 \otimes N \to M_2 \otimes N$$

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ that is full, faithful and injective on objects is called an embedding.

Example. We will revisit this example later, as it is the main motivation of this paper: a *cohomology* can be thought of as a functor $H : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is a category of topological spaces, and $\mathcal{D}$ is the category of abelian groups.

Example. Another very important and useful functor is the *presheaf* on a given category $\mathcal{C}$, which is a functor $F : \mathcal{C}^{op} \to \textbf{Set}$. To make this concrete, consider $\mathcal{C} = X$, where X is a topological space. It can quickly be checked that X forms a category where the open sets U are objects of this category, and embeddings f are the arrows. For each open set U, $F(U)$ is a set and for each $f : V \hookrightarrow U$, $F(f) = res_{V,U} : F(U) \to F(V)$, a restriction map in **Set**. More generally, we can define presheaves for any category $\mathcal{C}$ such that these two conditions hold.

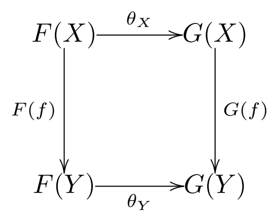
Example. "Is there a category of categories?" is a natural question to ask when learning about category theory. The answer is yes, absolutely. We denote this category by **Cat**. The objects of **Cat** are categories, and the arrows are functors between categories.

Another natural question to we ask, "Are there maps between functors?" The answer is: yes.

Definition. For categories $\mathcal{C}, \mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\theta : F \to G$ is a family of arrows in $\mathcal{D}$ indexed by objects $X \in \text{Ob}(\mathcal{C})$

$$\{\theta_X : F(X) \to G(X) | X \in \text{Ob}(\mathcal{C})\}$$

such that, for any $f \in \mathcal{C}(X, Y)$, there is $\theta_Y \circ F(f) = G(f) \circ \theta_X$ such that



commutes. For a given a natural transformation $\theta : F \to G$, the $\mathcal{D}$-arrow, $\theta_X : F(X) \to G(X)$ is called the *component of* θ at X.

Definition. A natural transformation $\theta : F \to G$ for functors $F, G : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism* if for every $X \in \text{Ob}(\mathcal{C})$, the component θ_X is an isomorphism in $\mathcal{D}$

Example. One example of a category of functors is the category of functors from a category $\mathcal{C}$ to **Set**, denoted $\mathbf{Set}^{\mathcal{C}}$. Among the special objects of this category are covariant *representable functors*, which are often denoted

$$\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set}$$

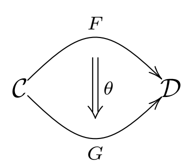
For each $f \in \mathcal{C}(X, Y)$, there is a natural transformation which is denoted by $\text{Hom}_{\mathcal{C}}(f, -) : \text{Hom}_{\mathcal{C}}(Y, -) \to \text{Hom}_{\mathcal{C}}(X, -)$. The component at X is defined as

$$(f : Y \to Z) \mapsto (g \circ f : X \to Z)$$

(This mapping should seem familiar if you've been following the examples!).

Example. Recalling our earlier example, we can define a category of pre-sheaves, denoted by $\mathbf{Set}^{\mathcal{C}^{op}}$.

Example. Although *2-categories*, and by extension *n-categories*, are far beyond the scope this paper, we can consider a category **2-Cat**, whose objects are categories, and for all objects $\mathcal{C}, \mathcal{D}$, a category $\mathbf{2-Cat}(\mathcal{C}, \mathcal{D})$ whose objects are functors $F, G : \mathcal{C} \to \mathcal{D}$, whose morphisms are natural transformations $\theta : F \Rightarrow G$. We represent this mapping with the following diagram:



There are additional properties to this category that the reader is encouraged to study on their own.

5. *A Small Detour to The Most Important Result in Category Theory: The Yoneda Lemma*

Although this result will not touched upon during the lecture, it is perhaps the most important result within category theory: the Yoneda lemma. Moreover, anyone reading these notes has been given enough machinery to both understand, prove and apply this result. First a definition.

Definition. The *Yoneda embedding* is a functor $\alpha : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$, which takes $X \in \text{Ob}(\mathcal{C})$ to a presheaf (sometimes in the literature, this is referred to as a contravariant representable functor),

$$\alpha(X) = \text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathbf{Set}$$

and which takes $f \in \mathcal{C}(X, Y)$ to the natural transformation

$$\alpha(f) = \mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{D}}(-, Y)$$

Definition. Given a natural transformation α and a presheaf F, we say that there is *naturality* in an object X for a category $\mathcal{C}$ when given any $f \in \mathcal{C}(X, Y)$, the following

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(Y), F) & \xrightarrow{\cong} & F(Y) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(f), F)} & & \Big\downarrow {\scriptstyle F(f)} \\
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(X), F) & \xrightarrow[\cong]{} & F(X)
\end{array}$$

commutes.

We say that there is *naturality in* F when given any $\theta : F \to G$, the following

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(X), F) & \xrightarrow{\cong} & F(X) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(f), \theta)} & & \Big\downarrow {\scriptstyle \theta_X} \\
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(Y), G) & \xrightarrow[\cong]{} & F(Y)
\end{array}$$

commutes.

Lemma 2. *(Yoneda Lemma) Let $\mathcal{C}$ be a locally small category. For any $X \in$ Ob $(\mathcal{C})$, and a presheaf $F \in \mathbf{Set}^{\mathcal{C}^{op}}$, there is an isomorphism*

$$Hom_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(X), F) \cong F(X)$$

which is natural in both F and X. That is to say, α, as defined above, is an embedding in the sense above.

Proof. (Sketch) The goal of this proof is to define an isomorphism $\eta_{X,F} : \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(\alpha(X), F) \to F(X)$. To do so, for each natural transformation $\theta : \alpha(X) \to F$, define

$$\eta_{X,F}(\theta) = \theta_X(1_X) \in F(X)$$

as $\theta_X : \mathcal{C}(X, X) \to F(X)$.
Then, given any $x \in F(X)$, we define a natural transformation $\theta_x : \alpha(X) \to F$ "component-wise" as follows: for any $Y \in$ Ob $(\mathcal{C})$, we define the component $(\theta_x)_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(Y)$ by

$$(\theta_x)_Y(f) = F(f)(x)$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. The reader is encouraged to verify that this satisfies the conditions of a natural transformation, defines an isomorphism and satisfies the naturality conditions. □

6. *What Is A Derived Functor, and How Do I Get One*

With the above tools, we can now turn our attention towards *derived functors* and see how they emerge in our study of group cohomology.

Definition. A locally small category $\mathcal{C}$ is an *additive* category if it has a zero object, finite products and coproducts, and each $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has an abelian structure such that the composition maps are bilinear.

Example. The category of abelian groups **Ab** is an additive category.

Example. The category of $R-$modules, denoted **R-mod**, whose objects are modules for a ring R and whose morphisms are module homomorphisms, is an additive category.

It should be clear from our earlier definitions that additive categories have zero morphisms. If $\mathcal{C}$ is an additive category, it also makes sense that one can "add" and "subtract" morphisms. In an additive category, morphisms f, g are "equal" if and only if $f - g = 0$.

Definition. An additive category $\mathcal{C}$ is an *abelian* category if kernels and cokernels exist in $\mathcal{C}$

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is *exact* if it preserves exact sequences from $\mathcal{C}$ into $\mathcal{D}$.

Definition. A *chain complex* $(M., d.)$ in an additive category $\mathcal{C}$ is a sequence of objects and morphisms

$$\ldots \overset{\partial_{i+2}}{\to} M_{i+1} \overset{\partial_{i+1}}{\to} M_i \overset{\partial_i}{\to} M_{i-1} \overset{\partial_{i-2}}{\to} \ldots$$

such that for all i, $\partial_i \circ \partial_{i+1} = 0$.
We define a *co-chain complex* $(M^{\cdot}, d^{\cdot})$ as a dual of a chain complex, and note the reversal of direction by moving our subscripts to superscripts.

Example. We denote category of co-chain complexes by $\mathfrak{C}(\mathcal{C})$. Objects in $\mathfrak{C}(\mathcal{C})$ are the cohcain complexes of a category $\mathcal{C}$. In this category, objects are cochain complexes $M^{\cdot}$ and $N^{\cdot}$, and morphisms $\alpha \in \mathrm{Hom}_{\mathfrak{C}(\mathcal{C})}(M^{\cdot}, N^{\cdot})$ are sequences of morphisms α_i in $\mathcal{C}$ such that

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{i-1} & \xrightarrow{\partial^{i-1}_{M^{\cdot}}} & M^i & \xrightarrow{\partial^{i}_{M^{\cdot}}} & M^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha^{i-1}} & & \downarrow{\scriptstyle \alpha^{i}} & & \downarrow{\scriptstyle \alpha^{i+1}} & & \\
\cdots & \longrightarrow & N^{i-1} & \xrightarrow{\partial^{i-1}_{N^{\cdot}}} & N^i & \xrightarrow{\partial^{i}_{N^{\cdot}}} & N^{i+1} & \longrightarrow & \cdots
\end{array}$$

commutes. That is to say, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{C})}(M^{\cdot}, N^{\cdot})$ consists of the above commutative diagrams.

Definition. A *resolution* of an object X is a complex whose cohomology is concentrated in degree 0, and is isomorphic to X.

Example. There is the trivial complex $\iota(X)$

$$\ldots \to 0 \to 0 \overset{\partial^1}{\to} X \overset{\partial^0}{\to} 0 \to 0 \to \ldots$$

Remark. Recall that the *nth cohomology* of a complex $(M^{\cdot}, \partial^{\cdot})$ measures its deviation from exactness at M^i:

$$H^n(M^{\cdot}) := ker(\partial^i)/im(\partial^{i-1})$$

Notation. By $\mathfrak{C}^+(\mathcal{C})$, we denote a full subcategory[3] of $\mathfrak{C}(\mathcal{C})$ determined by complexes $M^{\cdot}$, which are complexes bounded below. In other words, $M^i = 0$ for all $i << 0$. Similarly, we denote by $\mathfrak{C}^-(\mathcal{C})$ a full subcategory of $\mathfrak{C}(\mathcal{C})$ determined by complexes $M^{\cdot}$ that are bounded above.

[3] this means for all pairs of objects in the subcategory, the morphisms in the subcategory agree with the morphisms in $\mathfrak{C}(\mathcal{C})$.

Remark. The morphisms ∂^i are the differentials of the complex.

Remark. As in keeping with the spirit of this lecture, the condition that $\partial^{i+1} \circ \partial^i = 0$ is equivalent to $im(\partial^i) \subseteq ker(\partial^{i+1})$. That is, a complex is exact at each M^i if $im(\partial^{i-1}) = ker(\partial^i)$

For a morphism $\alpha : M^\cdot \to N^\cdot$ of cochain complexes, it is natural to look at a morphism induced in cohomology, which we denote by

$$H^\cdot(\alpha) : H^\cdot(M^\cdot) \to H^\cdot(N^\cdot)$$

Definition. A morphism $\alpha : M^\cdot \to N^\cdot$ of cochain complexes is a *quasi-isomorphism* if it induces an isomorphism in cohomology.

Definition. Let $\mathcal{C}$ be an abelian category. An object $P \in \text{Ob}(\mathcal{C})$ is *projective* if the functor $\text{Hom}_{\mathcal{C}}(P, -)$ is exact and an object $Q \in \text{Ob}(\mathcal{C})$ is *injective* if the functor $\text{Hom}_{\mathcal{C}}(-, Q)$ is exact.

Definition. We define a *homotopy* h between morphisms $\alpha, \beta : M^\cdot \to N^\cdot$ by a collection of morphisms $h^n : M^n \to N^n$ such that for all n

$$\beta^n - \alpha^n = \partial_{N^\cdot}^{n-1} \circ h^n + h^{n+1} \circ \partial_{M^\cdot}^n$$

If this condition is satisfied, we say that α is homotopic to β, and denote this by $\alpha \sim \beta$

Definition. Let $\mathcal{C}$ be an abelian category. We define the *homotopy category* $K(\mathcal{C})$ of cochain complexes in $\mathcal{C}$ as the category whose cochain complexes in $\mathcal{C}$ and whose morphisms are

$$Hom_{K(\mathcal{C})}(M^\cdot, N^\cdot) := Hom_{\mathfrak{C}(\mathcal{C})}(M^\cdot, N^\cdot)/\sim$$

where $\sim$ is a homotopy relation. Checking that this satisfies the definition of a category is well beyond the scope of this seminar, but enough machinery[4] has been provided to the reader to verify that it indeed is a category.

Remark. We will denote the subcategory of projective objects in an abelian category $\mathcal{C}$ by **Proj**. Similarly, we will denote the subcategory of injective objects in an abelian category by **Inj**.

Remark. We say that an abelian category $\mathcal{C}$ has *enough injectives* if for all $X \in \text{Ob}(\mathcal{C})$, there exists a monomorphism $f : X \to I$, where I is an injective object of $\mathcal{C}$. This would be a generalized injective module. Dually, an abelian category has *enough projectives* if for all $X \in \text{Ob}(\mathcal{C})$, there exists an epimorphism from $P \to X$, where P is a projective object.

Definition. Let X be an object of an abelian category $\mathcal{C}$.
A *projective resolution* of X is a quasi-isomorphism $P^\cdot \to \iota(X)$ where $P^\cdot$ is a complex in $\mathfrak{C}^-(\textbf{Proj})$.
Similarly, an *injective resolution* is a quasi-isomorphism $\iota(X) \to Q^\cdot$, where $Q^\cdot$ is a complex in $\mathfrak{C}^+(\textbf{Inj})$

Example. Revisiting an earlier example. For every integer n, the mapping

$$M^\cdot \mapsto H^n(M^\cdot)$$

defines an additive covariant functor $\mathfrak{C}(\mathcal{C}) \to \mathcal{C}$.In other words, each H^n induces in a functorial way homomorphisms of abelian groups

$$\text{Hom}_{\mathfrak{C}(\mathcal{C})}(M^\cdot, N^\cdot) \to \text{Hom}(H^n(M^\cdot), H^n(N^\cdot))$$

for all complexes $M^\cdot, N^\cdot$. This immediately follows from the commutativity requirement of morphisms α in $\mathfrak{C}(\mathcal{C})$. The composition $\partial_{N^\cdot}^i \circ \alpha^i = \alpha^{i+1} \circ \partial_{M^\cdot}^i$ is 0 on $ker(\partial_{M^\cdot}^i)$ by the commutativity of the square. Thus the restriction of α^i to $ker(\partial_{M^\cdot}^i)$ factors through $ker(\partial_{N^\cdot}^i)$

[4]The only machinery that needs to be given to complete the verification is that a homotopy relation $\sim$ respects composition. That is, $Hom_{\mathfrak{C}(\mathcal{C})}(M^\cdot, N^\cdot) \times Hom_{\mathfrak{C}(\mathcal{C})}(N^\cdot, S^\cdot) \to Hom_{\mathfrak{C}(\mathcal{C})}(M^\cdot, S^\cdot)$ descends to $Hom_{K(\mathcal{C})}(M^\cdot, N^\cdot) \times Hom_{K(\mathcal{C})}(N^\cdot, S^\cdot) \to Hom_{K(\mathcal{C})}(M^\cdot, S^\cdot)$

Definition. If abelian categories $\mathcal{C}, \mathcal{D}$ have enough projectives, then *left-derived functor* of $F : \mathcal{C} \to \mathcal{D}$, denoted LF, is the composition

$$K^-(\mathbf{Proj}(\mathcal{C})) \xrightarrow{\mathfrak{I}_\mathcal{C}} K^-(\mathcal{C}) \xrightarrow{K(F)} K^-(\mathcal{D}) \xrightarrow{P_\mathcal{D}} K^-(\mathbf{Proj}(\mathcal{D}))$$
$$LF : K^-(\mathbf{Proj}(\mathcal{C})) \to K^-(\mathbf{Proj}(\mathcal{D}))$$

where $\mathfrak{I}_\mathcal{C}$ is the inclusion functor. $P_\mathcal{D}$ is the resolution functor, associating every bounded above complex to any projective resolution $P_\mathcal{D}(M^\cdot)$ of $M^\cdot$ and with every morphism $\alpha : M^\cdot \to N^\cdot$ in $\mathfrak{C}^-(\mathcal{C})$, the morphism $P_\mathcal{D}(\alpha)$ in $K^-(P_\mathcal{D})$ lifting α. This is to say that $P_\mathcal{D}(\alpha)$ is the homotopy class of a homotopy lift of α.
Moreover, LF satisfies the following universal property. There is a natural transformation $\eta : LF \circ P_\mathcal{C} \to P_\mathcal{D} \circ K(F)$, and for every functor $G : K^-(\mathbf{Proj}(\mathcal{C})) \to K^-(\mathbf{Proj}(\mathcal{D}))$ there is a unique natural transformation $\beta : G \to LF$ which induces a factorization of $\gamma : G \circ P_\mathcal{C} \to LF \circ P_\mathcal{C} \to P_\mathcal{D} \circ K(F)$.

Similarly, we define *right derived functors* RF when $\mathcal{C}, \mathcal{D}$ have enough injectives. $F : \mathcal{C} \to \mathcal{D}$, denoted RF, is the composition

$$K^+(\mathbf{Inj}(\mathcal{C})) \xrightarrow{\mathfrak{I}_\mathcal{C}} K^+(\mathcal{C}) \xrightarrow{K(F)} K^+(\mathcal{D}) \xrightarrow{I_\mathcal{D}} K^+(\mathbf{Inj}(\mathcal{D}))$$
$$RF : K^+(\mathbf{Inj}(\mathcal{C})) \to K^+(\mathbf{Inj}(\mathcal{D}))$$

Example. Let G be a group and consider modules M over the group ring $\mathbb{Z}[G]$. These modules form an abelian category with enough injectives.
Let M^G denote the subgroup of M consisting of all elements fixed by G. This is a left exact functor and its right derived functors are the group cohomology functors written as $H^i(G, M)$.
This is how we can look at group cohomology as a way of studying groups using a sequence of functors H^n.

Example. (The **Tor** functor) Another example, which we have seen in other lectures, are the derived functors of the tensor product.
We can measure the exactness (or rather the failure of the exactness) of the functor $- \otimes_R N$ by another endofunctor on **R-mod**, denoted by $Tor_1^R(-, N)$. We extend this notion to a functor on sequences, and define a new functor by the homology of the complex $M. \otimes_R N$ by

$$Tor_i^R(M, N) := H_i(M. \otimes N)$$

One property about this functor that should be noted, if $Tor_1^R(M, N) = 0$ for all modules M, then $Tor_i^R(M, N) = 0$ for all $i > 0$ for all M.