A Tour of Games and Numbers

Alexander Berenbeim

Today

Outline

1. A Tour of Games and the Surreal Numbers
   - Games
   - Numbers
   - A Quick Detour Through Genetic Functions
   - Sign sequence lemma
   - Present work
   - Partial orderings on numbers

Combinatorial Games

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- H is a subposition of G if there exists a sequence of consecutive (not necessarily alternating) moves leading from G to H

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- G is transfinite if it is not necessarily finite;
- G is loopy if it is not necessarily loopfree;
- G is partizan if it is not necessarily impartial;

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- We say $G \geq 0$ if there is a winning strategy for the left; $G \leq 0$ if there is a winning strategy for the right, and $G \| 0$, or G is fuzzy if there is a winning strategy for the first player, and $G = 0$ if there is a winning strategy for the second player.

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- We define the negation of a game $G \in \text{PG}$ by

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- PG is a partially ordered abelian group, i.e. if $G \geq H$, then for any K, $G + K \geq H + K$.
- Furthermore, Lurie proved that PG is a universal embedding object in the sense that every ordered abelian group embeds into PG.

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- We can inductively construct $\mathrm{PG} = \bigcup_{\alpha \in \mathrm{On}} \mathrm{PG}_\alpha$ with

$$PG_0 = \{0\}$$

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- Given this construction, one can readily encode the ordinals as games, (or more precisely), as numbers as follows:

$$\alpha = \alpha|\emptyset = \{\alpha^L\} \mid \{\}.$$

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Theorem

(Fundamental Existence Theorem) For all sets of numbers $F < G$, there is a unique c of minimal length such that $F < c < G$.

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- $\beta_\mu(a) = 0$ implies that $\mu = \max \phi a$ or $\mu \geq \phi a$.
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- We can see this agrees with the previous construction of the surreal numbers, as $\iota a = \bigoplus_{\mu \in \phi a} \alpha_\mu \oplus \beta_\mu$

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- The genetic definition of multiplication is as follows:

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- Finally, let $\langle\rangle = 0$, and $\langle a_1, \ldots, a_n, a_{n+1} \rangle = \langle a_1, \ldots, a_n \rangle^\circ a_{n+1}$.Now define $a^{-1} = F|G$, where F consists of $\langle a_1, \ldots, a_n \rangle$ where the number of $a_i \in L_a$ is even and G where the number of $a_i \in L_a$ is odd.

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- These two respective normal forms can be used to define Krull valuations $-\ell$ on No, where $\ell : \text{No}^\times \to \text{No}$ where $\ell(a) = \max\{a_i \in \text{No} \mid r_i \neq 0\}$.

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- Given $a \in \mathrm{No}_{>0}$, define $a^\flat$ to be the surreal number

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- Similarly, given $a \in \mathrm{No}_{<0}$, define $a^\sharp$ to be the surreal number attained by omitting the first $\ominus$ sign.

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Given a surreal number $a = \sum_{i \in \nu a} \omega^{a_i} r_i$ in normal form, we define the **reduced sequence** $(a_i^o | i \in \nu a)$ by omitting $\ominus$ from the following sign sequences:

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- if i is a successor, $a_{i-1} \frown \ominus \sqsubset a_i$ and if r_{i-1} is not a dyadic rational, then omit $\ominus$ after a_{i-1} in a_i.

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Theorem

Given $a = (\langle \alpha_i, \beta_i \rangle)_{i \in \phi a}$, *then* ω^a *has the sign sequence*

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Proof.

This follows directly from $\iota(a \frown b) = \iota(a) \oplus \iota(b)$, and by induction on νa. □

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then $|\beta| \leq |\text{lub}_{\alpha \in \beta}[\iota(a_\alpha)\omega]|$. (This result refers to the least upper bound of ordinals on the right hand side, and cardinalities on the left hand side).

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$\iota(a) = \iota([a]) + \iota(a - [a])$ where $[a]$ denotes the natural number part of a;

- For $a, b \in \mathbb{R}$, $\iota(ab) \leq \iota(a)\iota(b)$;
- Let $x, y \in \mathrm{No}$, and $0 < r \in \mathbb{R}$, then we have:

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- *Finally, set $\zeta_\mu = \sum_{N\xi_\mu} \zeta_{\mu,i}$.*

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- If we can prove the product lemma for the case where $a > b$ so that $x = \omega^a + \omega^b$, and $y = \omega^c$, that $\iota(xy) \leq \iota(x)\iota(y)$, then by induction this can prove the product lemam in general.

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- In turn, since $a^+ = a^{o+}$, it suffices to show that if $\chi(ab) \leq \chi(a)\chi(b)$, as the principle obstruction is reduction.

Current work product lemma restated

Proposition

Let $a > b$ and c be arbitrary surreal numbers. Consider $(b+c)^o$ the reduction of $(b+c)$ with respect to $a+c$ and b^o the reduction of b with respect to a. Then $\chi((b+c)^o) \leq \chi((b)^o)\chi(c)$ implies $\iota(\omega((b+c)^o)) \leq \iota(b^o)\iota(c)$.

Proof.

Recall from Fact ??, that $(a+b)^+ \leq a^+ + b^+$, and $\iota\omega(x) = \omega^{x^+}\chi x$. Since reduction only eliminates $\ominus$ symbols, we find that

$$(b+c)^{o+} = (b+c)^+$$

. □

Current work product lemma restated

Proof.

so if $\chi((b+c)^o \leq \chi((b^o))\chi(c)$, we have:

$$\begin{aligned}
\iota((b+c)^o) &= \omega^{(b+c)^{o+}}\chi((b+c)^o) \\
&= \omega^{(b+c)^+}\chi((b+c)^o) \\
&\leq \omega^{b^+}\omega^{c^+}\chi((b+c)^o) \\
&\leq \omega^{b^+}\omega^{c^+}\chi(b^o)\chi(c) \\
&= \omega^{b^+}\chi(b^o)\omega^{c^+}\chi(c) \\
&= \omega^{b^{o+}}\chi(b^o)\omega^{c^+}\chi(c) \\
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Moreover, these constructions are functorial in the sense that they can be defined as enriched categories over the category of the ordinals.

- Results for $\mathbb{R}_{an}$, valued fields, and real differentiable fields will likely follow, and likely correspond to λ and κ numbers that were omitted from this talk.

Simplicity

Proposition

$(\mathrm{No}, <_s^o)$ *is a separative partial order under reverse inclusion.*

Proof.

It is immediate that No is partially ordered by $<_s$, and so No will also be partially ordered by the opposite $<_s^o$, with top element 0.
Now suppose $a, b \in \mathrm{No}$ have tree rank α, β respectively and are such that $a \not\leq_s^o b$. Then $b \not\sqsubset a$, and so either $a \sqsubset b$ or $a \perp b$.
If $a \sqsubset b$, then there is some $x \in \{-, +\}$ such that $a \frown x \sqsubset b$. Let $y = \neg x$ (i.e. $\neg - = +$ and $\neg + = -$), and consider $c = a \frown y$. Then $a \sqsubset c$, hence $c \leq_s^o a$ and $c \perp b$ as desired. If $a \perp b$, then we may take $a = c$. □