Foundations for the Analysis of Surreal-valued genetic functions

AMS Special Section for Topics in Model Theory

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Outline

Background and Motivation

- Foundations: We start by using NBG, since we want to reason about differences between Classes and Sets, and quantify over sets, but in practice we can really use ZFC (or ZF + DGC).
- Analysis: following Sikorski, the sequences are On-length sequences
- Surreal-valued genetic functions: will take the rest of the talk to describe
- Motivation: van den Dries and Ehrlich showed in 2001 that truncating the binary tree of surreal numbers No at heights of ϵ_γ yields natural models of $\mathbb{R}_{\text{exp}}$, in turn establish the surreal numbers as a universal object.
- Two languages to consider $\mathcal{L} = \langle < \rangle \cup \mathcal{G}$ and $\mathcal{L}^s = \langle <, <_s \rangle \cup \mathcal{G}$.
- Want to use facts about the reducts of structures of $\mathcal{L}^s$ to $\mathcal{L}$.

Partizan Games and Disjunctive Game Compounds

- A combinatorial game G is a two player game, with players conventionally called Left and Right, who play alternately, and whose moves affect the position of the game according to rules.
- Games are partizan whenever these rules distinguish available moves to Left and Right players. Otherwise they are impartial
- If G and H are combinatorial games, H is a Left option of G whenever Left can move from G to H. Let L_G denote the set of all available direct moves for Left (and similarly R_G. Let G^L denote a generic Left move (and similarly G^R).
- We can form new games using the recursive disjunctive compound

$$G + H = \left\{G + H^L, G^L + H\right\} \Big| \left\{G + H^R, G^R + H\right\}.$$

Fundamental Examples

- (Endgame) $0 \equiv \{\} | \{\}$
- (Pos) $1 \equiv \{0\} | \{\}$
- (Neg) $-1 \equiv \{\} | \{0\}$
- (Fuz) $* \equiv \{0\} | \{0\}$

Numbers, Cuts, and Games

- Given an ordered space $(X, <)$, a Conway cut $(L|R)$ of X arises when $L \cup R \subset X$ and $L < R$.
- A Cuesta-Dutari cut is a Conway cut that partitions the ordered space X.
- The canonical realization of a Conway cut is denoted by $L|R$, i.e. the minimal set rank number c satisfying $L < c < R$.
- We can build up a lexicographically, partially well-ordered binary tree such that each level is a canonical realization of the cuts of the level below (modulo details of the limit case).
- We can think of $|$ as a mapping from the Class of Conway cuts $\mathfrak{C}$ to the Class of the surreal numbers, sending cuts (F, G) to the unique minimal set-theoretic ranked number c such that $F < c < G$ and for all x such that $F < x < G$, we have $c \leq_s x$.

Numbers and functions

- Surreal numbers are linearly ordered and definable wrt a canonical, genetic construction, with well-definedness as a consequence of the unique minimal realization of the cut generated by the predecessors sets $a^o = L_a \cup R_a$
- This genetic construction carries over via transfinite induction to define polynomials, and ultimately surreal-valued genetic functions
- These include numerous functions of interest to model theorists, such as exp and ω, as well as log and the λ function that identifies log-atomic numbers
- Importantly, each number has a corresponding sign sequence, indicating its lexicographical ordering and its complexity (we'll return to this later)

Motivating Genetic Functions (and the research)

- The disjunctive game compound $+$ is an order-preserving abelian group operation.
- We can recursively define a suitable notion of multiplication on the numbers by

$$ab = \{a^L b + ab^L - a^L b^L, a^R b + ab^R - a^R b^R\}|$$

$$\{a^L b + ab^R - a^L b^R, a^R b + ab^L - a^R b^L\}.$$

- This is another example of a recursively definable function with the uniformity propery (a la Gonshor)
- Functions are well-defined with respect to cofinality on the option sets

Adjoining New Function Symbols

Let v, w denote indeterminates, and let $f : \text{No} \to \text{No}$ be a function symbol, and suppose S is a set of genetic functions that have already been defined. Then

1. We form the Ring $K := \text{No}[\{g(v), g(w) \mid g \in S \cup \{f\}\}]$, where S is a set closed under composition consisting of previously defined genetic functions on one variable.
2. We obtain a Class

$$S(v, w) = \{c_1 + c_2 h(c_3 x + c_4) \colon c_1, c_2, c_3, c_4 \in K, h \in S\}.$$

3. We then form Ring $R(v, w) := \text{No}[S(v, w)]_{P_S}$, where P_S is the cone of strictly positive polynomials with function from S.

Amendments to Rubinstein-Salzedo and Swaminathan

We want to choose sets $L_f(v, w)$ and $R_f(v, w)$ from $R(v, w)$ such that the order condition and cofinality condition will hold:

- Fix an $x \in \text{No}$, and suppose $f(y)$ has already been defined for all $y \in L_x \cup R_x$, substitute v with x^L and w with x^R in $R(v, w)$
  - (Order Condition) all $x^L, x^{L'} \in L_x$ and $x^R, x^{R'} \in R_x$, and $f^L \in L_f(x^L, x^R)$ and $f^R \in R_f(x^{L'}, x^{R'})$ we have $f^L(x) < f^R(x)$, and
  - (Cofinality Condition)

$$\forall x, y, z \in \text{No}((y < x < z) \rightarrow$$

$$L_f(y, z)[x] < f(x) < R_f(y, z)[x].$$

  - Once f is defined over No, we prove that the cofinality condition holds, via (double) induction with respect to the natural sum of the lengths of the arguments and generation.

Amendments to Rubinstein-Salzedo and Swaminathan

Finally, set

$$f(x) := \{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^L(x) \colon f^L \in L_f(x^L, x^R)\}\} |$$

$$\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^R(x) \colon f^R \in R_f(x^L, x^R)\}\}$$

What this means in practice

- Genetic functions are defined pointwise with respect to a Conway cut whose Left and Right options are defined with respect to substituting in the Left predecessors of x for u and the Right predecessors of x for v.
- However, a cofinality condition must hold, and so sets must form a generic cut in the sense that for all $y < x < z$ we can substitute in y/u and z/v so that $f^L(x; y, z) < f^R(x; y, z)$ as f^L and f^R vary.
- These *sets* have a fixed order type corresponding to the terms formed in the polynomial ring $R(u, v)$ localized at the positive cone P_S, although in practice, these size of these sets grows with the complexity of the argument.
- The base case of each function is always defined by the constants appearing in the Left and Right option sets, and the application of previously defined genetic functions at 0.
- When analyzing the complexity, we can induct on the complexity of our term sets and use pseudo-absolute values to bound our functions.

Example: exp and ω

- Let $[x]_n = \sum\limits_{i \leq n} \frac{1}{i!} x^i$.
- Gonshor records Kruskals genetic definition of exp (p145 of Gonshor):

$$\exp(x) = \{0, \exp(x^L)[x - x^L]_n, \exp(x^R)[x - x^R]_{2n+1}\}$$

$$\{\frac{1}{[x^R - x]_n} \exp(x^R), \frac{1}{[x^L - x]_{2n+1}} \exp(x^L)\}$$

- Following Conway, we can define ω:

$$\omega(x) = \left\{0, n\omega(x^L)\right\} \Big| \left\{\frac{1}{2^n}\omega(x^R)\right\}$$

which agrees with the ordinal valued ω function when restricted to ordinals.

Pseudo-absolute values

Let $\varsigma : S_1 \to S_2$ be a map between two semi-rings. We say ς is a **pseudo-absolute value if the following holds:**

1. $\varsigma(x) = 0 \iff x = 0$;
2. $\varsigma(xy) \leq \varsigma(x)\varsigma(y)$;
3. $\varsigma(x + y) \leq \varsigma(x) + \varsigma(y)$

- (Returning to motivation) We know that $\iota(a + b) \leq \iota(a) + \iota(b)$, but no proof for $\iota(ab) \leq \iota(a)\iota(b)$ exists
- (van den Dries-Ehrlich 2001) $\iota(\mathrm{ab}) \leq \omega[\iota(\mathrm{a})]^2[\iota(\mathrm{b})]^2$

Defining $\surd$

Let $\alpha_1, \alpha_2 \in \mathrm{On}$ have Cantor normal form $\sum_{j \in n_i} \omega^{\alpha_{i,j}}$ for $i = 1, 2$.

Then say $\alpha_1 \sim_\Gamma \alpha_2$ if and only if $\alpha_{1,0} = \alpha_{2,0}$.

- $\sim_\Gamma$ is an equivalence relation put on ordinals which extend to surreal number a via the equivalence Class $[\iota a]_{\sim_\Gamma}$.
- Each of these equivalence Classes has a *simplest* non-negative/ordinal element.
- The simplest element is the Γ-ordinal ω^α such that $\omega^\alpha \sim_\Gamma \iota a$.
- We denote this simplest element by $\surd\iota a$, and for general $\alpha \in \mathrm{On}$, by $\surd\alpha$.
- We can extend this equivalence relation to all No by collapsing the Levels of No, so in turn, the simplest element of $[\iota a]_{\sim_\Gamma}$ is also the minimal non-negative element of $[a]_{\sim_\Gamma}$.
- $\surd$ is a Class function from the Class of ordinals to the Class of Γ-ordinals.

Veblen hierarchy

- A normal ordinal valued function φ_0 is any continuous (with respect to the order topology) strictly increasing ordinal valued function.
- Given a normal function φ_0, the Veblen functions with respect to φ_0 are the sequence of functions $\langle \varphi_\alpha : \alpha \in \mathrm{On} \rangle$ such that each φ_α enumerates the common fixed points of φ_β for every $\beta \in \alpha$.
- The Veblen hierarchy is the Class of functions $\langle \varphi_\alpha : \alpha \in \mathrm{On} \rangle$ generated by $\varphi_0(x) = \omega^x$.
- Finally, we have the following ordering on the Veblen hierarchy:

$$\varphi_\alpha(\beta) < \varphi_\gamma(\delta) \iff$$

$$(\alpha = \gamma \wedge \beta < \delta) \vee (\alpha < \gamma \wedge \beta < \varphi_\gamma(\delta)) \vee (\alpha > \gamma \wedge \varphi_\alpha(\beta) < \delta))$$

Veblen hierarchy (ctd)

Recall

$$\omega(x) = \left\{0, \omega(x^L)n\right\} \Big| \left\{\omega(x^R)2^{-n}\right\}$$

and

$$\epsilon(x) = \left\{0, \omega^{(n)}(0), \omega^{(n)}(\epsilon(x^L)+1)\right\} \Big| \left\{\omega^{(n)}(\epsilon(x^R)-1)\right\}$$

+ Because ω is a genetic function, it is immediate that every Veblen function is a genetic function

- In fact, we could show that the construction of g in GFPT given $\varphi_0(x) = \omega(x)$ is equicofinal with the construction of $\epsilon(x)$.
- Our primary motivation here is to identify for every $g \in \mathscr{G}$, the least α such that for all $\gamma \in \mathrm{On}$, if $x \in \mathrm{No}(\gamma)$ then $g(x) \in \mathrm{No}(\varphi_\alpha(\gamma))$.

Gonshor Fixed Point Theorem

Theorem (Gonshor Fixed Point)

Suppose $f : \mathrm{No} \to \mathrm{No}$ satisfies the following properties:

1. *For all $a \in \mathrm{No}$, $f(a)$ is a power of ω;*
2. *$a < b \Rightarrow f(a) < f(b)$;*
3. *There are two fixed sets C and D such that whenever $a = G|H$ where G contains no maximum and H contains no minimum, then $f(a) = (C \cup f(G))|(D \cup f(H))$.*

Then the function g defined by

$$g(b) := \left\{ f^{(n)}(C), f^{(n)}(2g(b^L)) \right\} \Big| \left\{ f^{(n)}(D), f^{(n)}(\frac{1}{2}g(b^R)) \right\}$$

is onto the set of all fixed points of f and satisfies the above hypotheses with respect to the sets $f^{(n)}(C)$ and $f^{(n)}(D)$, where $f^{(n)}$ denotes the n^{th} iterate of f. Furthermore, there is a On-length family of functions f_α satisfying all three conditions, such that $f_0 = f$ and for $\alpha > 0$, f_α is onto the set of all common fixed points of f_β for $\beta \in \alpha$ and satisfies condition (iii) with respect to the sets $h(C)$ and $h(D)$ where h runs through all finite compositions of f_β for $\beta \in \alpha$.

Partial Veblen Rank

We inductively define the notion of partial Veblen rank as follows: Fix $\gamma \in \mathrm{On}$, $VR(g, \gamma)$ is defined as follows:

- $VR(g, \gamma) \geq 0$;
- $VR(g, \gamma) \geq \lambda$ for limit ordinals λ if and only if $VR(g, \gamma) \geq \beta$ for all $\beta \in \lambda$;
- $VR(g, \gamma) \geq \alpha + 1$ if and only if there is an $x \in \mathrm{No}(\epsilon_\gamma)$ such that $\sqrt{g(x)} \geq \varphi_{\alpha+1}(\gamma)$.

Full Veblen Rank

- We say $VR(g, \gamma) = \alpha$ whenever $VR(g, \gamma) \geq \alpha$ and $VR(g, \gamma) \not\geq \alpha + 1$, i.e. α is the least ordinal such that for all $x \in \mathrm{No}(\varphi_1(\gamma))$, $g(x) \in \mathrm{No}(\varphi_{\alpha+1}(\gamma))$.
- We then define the Veblen rank of g by $VR(g) := \bigcup_{\gamma \in \mathrm{On}} VR(g, \gamma)$.
- We can extend this definition to $g : \mathrm{No}^n \to \mathrm{No}$ by noting that $\iota(\bar{x})$ is the Hessenberg sum of the lengths of the components, so we can interpret $\mathrm{No}^n(\epsilon_\gamma)$ as the initial subset of No^n consisting of n-tuples of branches whose Hessenberg sum is less than ϵ_γ.
- Whenever $VR(g) \geq \alpha$ for all $\alpha \in \mathrm{On}$, rather than denote this by saying the rank is ∞, we indicate this by saying the rank is On.
- As an aside, we can extend the notion of Veblen rank from entire genetic functions, to those that are defined on convex intervals of surreal numbers, like all positive surreal numbers in the case of log.

Main Theorem

Theorem (Main Theorem (B., '21))

Every genetic function g has Veblen rank in On*, i.e.*
$\exists\gamma \in \mathrm{On}\forall\beta \in \mathrm{On}(\gamma \in \beta \Rightarrow VR(g,\beta) \leq VR(g,\gamma)$*.*

This proof is built on the following:

Lemma (B, '21)

For all surreal-valued genetic functions f, g,

- $VR(f + g) \leq \max\{VR(f), VR(g)\}$
- $VR(fg) \leq \max\{VR(f), VR(g)\}$
- $VR(f \circ g) \leq \max\{VR(f), VR(g)\}$
- *For a set S of genetic functions, and any term t generated by $\mathcal{L}_{oring} \cup S$,*

$$VR(t^n) \leq VR(t) \leq \sup\{VR(g)\colon g \in S\}$$

Examples

The following have zero Veblen rank:

- Identity
- Addition
- Negation
- Multiplication
- exp
- ω
- log

Additionally, each Veblen function $\varphi_\alpha(x)$ has Veblen rank α (B, '21)

$VR(\kappa) = 1$ (B., '21)

- The κ numbers are the simplest elements in their respective exp-log class.
- The genetic definition is given by

$$\kappa(x) := \left\{\exp^{(n)}(0), \exp^{(n)}(\kappa(x^L))\right\} \Big| \left\{\log^{(n)}(\kappa(x^R))\right\}$$

- One can check that $\kappa(1) = \epsilon_0 = \varphi_1(0)$, and so we have a witness to $VR(\kappa) \geq 1$.
- One checks by induction that $VR(\kappa, \gamma) \leq 1$ on all γ, while using some bounds on the complexity of exp and log established by van den Dries and Ehrlich.
- In particular, one checks for all $x \in \mathrm{No}(\epsilon_\gamma)$ that $\sqrt{(\kappa(x))} < \varphi_2(\gamma)$, which follows using the aforementioned inequalities to show that

$$\sqrt{(\kappa(x))} \leq \varphi_1(\iota x) < \varphi_1(\varphi_1(\gamma)) < \varphi_2(\gamma)$$

$VR(\lambda) = 1$ (B., '21)

Definition

Let $a \in \mathrm{No}^{>0}_{>0}$, i.e. a is a positive infinite surreal number. We say a is **log-atomic** if for all $n \in \mathbb{N}$, there is a $b_n \in \mathrm{No}$ such that for the n-fold iterate of log we have

$$\log^{(n)}(a) = \omega^{b_n}.$$

We denote the class of log-atomic numbers by $\mathbb{L}$.

- Then defining

$$\lambda(x) = \left\{ m, \exp^{(n)}(n \cdot \log^{(n)} \lambda(x^L) \right\} \Big| \left\{ \exp^{(n)} \left(\frac{1}{m} \log^{(n)} \lambda(x^R) \right) \right\}$$

  one sees the λ numbers correspond to the log-atomic numbers
- following applications of the aforementioned inequalities in the previous slide, we also establish that $VR(\lambda) = 1$.

Questions and Future directions

- Can similar work be done for characteristic p cases? p-adic cases?
- Generalizing work on homogeneous and model-complete theories? (when are we guaranteed to get initial embeddings)
- What about realization in *exotic* set theories?

Thank You

THANK YOU!