Mekler Constructions and Preservation of Stability

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2019-04-17

Outline

Main Idea

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- $G(\Gamma)$ is freely generated in the variety of 2-nilpotent groups of exponent p by the vertices of Γ by imposing the condition that two generators commute if and only if they share an edge in $Γ$
- Mekler's construction is a functorial construction that preserves stability;pause it also preserves NIP, k-dependence, and NTP_2.
- In this talk, I'll focus on the preservation of k-dependence.

Some Group Theory Reminders

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- The exponent of a group G is the least common multiple of the orders of all elements of the group.
- G is nilpotent if there is a central series terminating with G, i.e. there is a series of normal subgroups such that

$$e = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

  with $G_{i+1}/G_i \leq Z(G/G_i)$, i.e. $[G, G_{i+1}] \leq G_i$
- G is nilpotent class n if n is the least n such that G has a central series length n
- Any nonabelian group G such that $G/Z(G)$ is abelian has nilpotency class 2 with central series

$$\{e\} \trianglelefteq Z(G) \trianglelefteq G$$

- Examples of 2-nilpotent groups include the Heisenberg group and the quaternions.

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1. *there are at least two vertices, and for any two distinct vertices x, y there is a vertex z not equal to x or y such that $E(x, z) \wedge \neg E(y, z)$*
2. *Γ is triangle and square free.*

Covers of nice graphs

Definition

Let Γ *be an infinite nice graph. A graph* $\Gamma^+ \supset \Gamma$ *as a subgraph is a* cover *of* Γ *if for all* $b \in \Gamma^+ \backslash \Gamma$, *one of the following holds:*

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A proper cover of a nice graph is never a nice graph.

Three Definable Equivalence Relations

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One can readily see that $g \equiv_Z h \Rightarrow g \approx h \Rightarrow g \sim h$

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We can partition all elements of G into the following 5 $\emptyset$ definable classes:

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4. *elements of type p;*

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5. *elements of type p-1.*

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Additionally, for all elements g of type p, the noncentral elements commuting are precisely $[g]_\sim$ and an element b of type 1^ν along with $[b]_\sim$.

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- Handles are definable from g up to \$$\sim$\$-equivalence
- Since Z(G) and $G/Z(G)$ are elementary abelian p-groups, we can view them as $\mathbb{F}_p$ vector spaces.
- Independence considered over some supergroup of $Z(G)$ is linear independence in terms of the corresponding $\mathbb{F}_p$ vector space.

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- A 1^ι transversal of G is a set of $X\iota$ representatives of $\sim$ classes of proper elements of type 1^ι which is maximally independent modul the subgroup generated by the type 1^ν elements and $Z(G)$;

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Lemma

For $G \models Th(G(\Gamma))$, and given a small tuple of variables $\bar{x} = \bar{x}^{\nu} \frown \bar{x}^{p} \frown \bar{x}^{\iota}$, there is a partial type $\Phi(\bar{x})$ such that for any typles $\bar{a}^{\nu}, \bar{a}^{p}, \bar{a}^{\iota}$ in G, we have $G \models \Phi(\bar{a}^{\nu}, \bar{a}^{p}, \bar{a}^{\iota})$ if and only if every element belongs to the appropriate type, and $\bar{a} = \bar{a}^{\nu} \frown \bar{a}^{p} \frown \bar{a}_{\iota}$ can be extended to a transversal of G.

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- For a nice graph Γ, there is an interpretation M such that for any $G \models Th(G(\Gamma))$, we have $M(G) = (V, R)$, where $V = \{g \in G \mid \text{g is of type } 1^{\nu}, g \notin Z(G)\}/\approx$ and $([g]_{\approx}, [h]_{\approx}) \in R \iff gh = hg$, is a model of $Th(Γ)$

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- For a nice graph Γ, there is an interpretation M such that for any $G \models Th(G(\Gamma))$, we have $M(G) = (V, R)$, where $V = \{g \in G \mid$ g is of type $1^\nu, g \notin Z(G)\}/\approx$ and $([g]_\approx, [h]_\approx) \in R \iff gh = hg$, is a model of $Th(Γ)$
- The full set of transversals produces a cover of a nice graph
- A transversal$X$ can be viewed as a cover of the nice graph given by elements of the type 1^ν in X, with the edge relation given by commutation(we may identify X^ν with the set of vertices in $M(G)$ by mapping $x \in X^\nu$ with 4[x]$_\approx$$)

MOAR (transversal facts)

Lemma

For saturated $G \models Th(G(\Gamma))$ *and* X *a transversal of* G*, there is a subgroup* $K_X \leq Z(G)$ *such that* $G = \langle X \rangle \times K_X$*. Letting* Y, Z *be two small subsets of* X *and* $\bar{h}_1, \bar{h}_2$ *be tuples in* K_X*, then if*

- *there is a bijection* $f : Y \to Z$ *respecting the* $1^\nu, p, 1^\iota$ *parts, the handles, and* $tp_M(Y^\nu) = tp_M(f(Y^\nu))$*;*
- $tp_{K_X}(\bar{h}_1) = tp_{K_X}(\bar{h}_2)$

Then there is an automorphism of G *coinciding with* f *on* Y *sending* $\bar{h}_1$ *to* $\bar{h}_2$

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- *there is a bijection $f : Y \to Z$ respecting the $1^\nu, p, 1^\iota$ parts, the handles, and $tp_M(Y^\nu) = tp_M(f(Y^\nu))$;*
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Lemma

For G and X above, we have $G' = \langle X \rangle'$. i.e. The choice of a transversal and an elementary abelian subgroup of the center in the decomposition of G can be made independently.

Last Transversal Proposition

Proposition

For $G \models Th(G(\Gamma))$, $G \models \pi(\bar{a}, \bar{b})$ if and only if we can extend $\bar{a}$ to a transversal X of G and find $H \subset Z(G)$ containing $\bar{b}$ linearly indepdent over G' so that $G = \langle X \rangle \times \langle H \rangle$.

k-independence property

Definition

A formula $\phi(x; y_1, \ldots, y_k)$ *has the k-independence property with respect to T if in some model there is a sequence* $(\bar{a}_{ß)_{i \in \omega}}$ *such that for every* $s \subset \omega^k$*, there is* b_s *such that*

$$\models \phi(b_s; \bar{a}_I) \iff I \in s$$

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If not, then $\phi(x; \bar{y})$ is k-dependent. T is k-dependent if it implies every formula is k-dependent T is strictly k-dependent if it is k-dependent but not (k-1)-independent.

Ordered k-partite hypergraphs

Definition

Fix $L^k_{opg} = \{R(\bar{x}), <, P_0(x), \ldots, P_{k-1}(x)\}$. *An ordered k-partite hypergraph is an* L^k_{opg} *structure* $\mathcal{A}$ *such that:*

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The random ordered k-partite hypergraph

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With $\mathcal{K}$ the Fraïssé class of all finite ordered k-partite graphs, the limit of $\mathcal{K}$ is the ordered k-partite hypergraph, which will be denoted $G_{k,p}$.

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- $(P_i(A), <) \models \mathrm{DLO}$ *for all* $i < k$;
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The random ordered k-partite hypergraph

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Let $O_{k,p}$ denote the reduct of $G_{k,p}$ to the language $L^k_{op} = \{<, P_0, \ldots, P_{k-1}\}$

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- For T complete and $\mathbb{M} \models T$, and any $k \in \omega$, TFAE

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- Since nice Γ is interpretable in $G(\Gamma)$, if $Th(G(\Gamma))$ is k-dependent, then $Th(\Gamma)$ is k-dependent.

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- By the choice of X, H, for each $i < k, \alpha \in \kappa$, there is some term $t_{i,\alpha} \in L_G$ and finite tuples $\bar{x}_{i,\alpha} \in X$, $\bar{h}_{i,\alpha} \in H$ such that $a_{i,\alpha} = t_{i,\alpha}(\bar{x}_{i,\alpha}, \bar{h}_{i,\alpha})$.

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- Since $\kappa > |L_G| + \aleph_0$, we pass to a subsequence of length κ for each $i < k$, so that we may assume $t_{i,\alpha} = t_i$ and $\bar{x}_{i,\alpha} = \bar{x}^{\nu}_{i,\alpha} {}^\frown \bar{x}^{p}_{i,\alpha} {}^\frown \bar{x}^{\iota}_{i,\alpha}$.

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- By our structural Ramsey theory facts, we let $\langle \bar{y}_g{}^\frown \bar{m}_g \mid g \in O_{k,p} \rangle$ be $O_{k,p}$ indiscerible in G based on $\langle \bar{x}_g{}^\frown \bar{h}_g \mid g \in O_{k,p} \rangle$.

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3. the set of all elements of G appearing in $\langle \bar{y}_g \mid g \in O_{k,p} \rangle$ is a partial transversal, hence it can be extended to a transversal Y of G
4. the set of all elements in G appearing in $\langle \bar{m}_g \mid g \in O_{k,p} \rangle$ is a set of elements in $Z(G)$ linearly independent over G', and thus can be extended to a linearly independent set M such that $G = \langle Y \rangle \times \langle M \rangle$

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- We now expand $O_{k,p}$ to $G_{k,p}$. Since ψ shatters $\langle \bar{y}_g \frown \bar{m}_g \rangle$, we can find $b \in G$ such that

$$G \models \psi(b; \overline{(\bar{y} \frown \bar{m})_{\bar{g}}}) \iff G_{k,p} \models R(\bar{g}), \forall g_i \in P_i$$

Overview of the Proof 4

- We write $b = s(\bar{z}, l)$ for some term $s \in L_G$, and some tuple $\bar{z} = \bar{z}^\nu \frown \bar{z}^p \frown \bar{z}^\iota \in Y$, and $\bar{l} \in M$, extending $\bar{z}^\nu$ if necessary so that $\bar{z}$ is closed under handles.

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- We now set $\$\theta(\mathrm{x}\hat{}\,/;\hat{}\,/)\mathrm{:=}\psi(\mathrm{s}(\mathrm{x}\hat{}\,/);\hat{}\,/)\$$

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- Among the consequences is that the set of all elements of G appearing $\bar{l}$ and $\langle \bar{l} \mid g \in G_{k,p} \rangle$ remains a subset of $Z(G)$ that is linearly independent over G', and hence can be extended to a linearly independent set L such that $G = \langle Z \rangle \times \langle L \rangle$.

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- We write $b = s(\bar{z}, l)$ for some term $s \in L_G$, and some tuple $\bar{z} = \bar{z}^{\nu} \frown \bar{z}^{p} \frown \bar{z}^{\iota} \in Y$, and $\bar{l} \in M$, extending $\bar{z}^{\nu}$ if necessary so that $\bar{z}$ is closed under handles.
- We now set $\theta(x^\frown l;^\frown l){:=}\psi(s(x^\frown l);^\frown l)$
- $G \models \theta(\bar{z}^\frown \bar{l}; \overline{(\bar{y}^\frown \bar{m})_{\bar{g}}) \iff G_{k,p} \models R(\bar{g}), \forall g_i \in P_i}$
- By structural Ramsey fact, we can find an $G_{k,p}$ indiscernible sequence $\langle \bar{z}_g \frown \bar{l}_g \mid g \in G_{k,p} \rangle$ over $\bar{z} \frown \bar{l}$ and based on $\langle \bar{y}_g^\frown \bar{m}_g \mid g \in G_{k,p} \rangle$ over $\frown$
- Among the consequences is that the set of all elements of G appearing $\bar{l}$ and $\langle \bar{l} \mid g \in G_{k,p} \rangle$ remains a subset of $Z(G)$ that is linearly independent over G', and hence can be extended to a linearly independent set L such that $G = \langle Z \rangle \times \langle L \rangle$.Additionally, the $G_{k,p}$ indiscernible sequence is $O_{k,p}$ indiscernible over $\emptyset$.

Overview of the Proof 5

- Since $Th(\Gamma)$ is k-dependent, it follows that $\langle \bar{z}_g^\nu \mid g \in G_{k,p} \rangle$ is $G_{k,p}$ indiscernible over $\bar{z}^\nu$ and $O_{k,p}$ indiscernible over $\emptyset$ in $M(G)$

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- Thus for all $\bar{g}, \bar{q} \in G_{k,p}$ such that $tp_{L^{k}_{op}}(\bar{g}) = tp_{L^{k}_{op}}(\bar{q})$, we have $tp_M(\bar{z}^{\nu}_{\bar{g}}/\bar{z}^{\nu}) = tp_M(\bar{z}^{\nu}_{\bar{q}}/\bar{z}^{\nu})$

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- By $O_{k,p}$ indiscernibility, and finiteness of $\bar{z}$, there is $\lambda_i \subset P_i$ for each $i < k$ such that for all $g \neq q \in P_i$, with $g, q > \lambda_i$, we have

$$\bar{z}^{p}_{g} \cap \bar{z}^{p} = \bar{z}^{p}_{q} \cap \bar{z}^{p} \wedge \bar{z}^{\iota}_{g} \cap \bar{z}^{\iota} = \bar{z}^{\iota}_{q} \cap \bar{z}^{\iota}$$

  with $\bar{z}_g \cap \bar{z}_q$ constant.

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  with $\bar{z}_g \cap \bar{z}_q$ constant.
- For $g_i, q_i > \lambda_i$, we get a mapping of $(\bar{z}_{\bar{g}}, \bar{z}) \mapsto (\bar{z}_{\bar{q}}, \bar{z})$ which preserves the positions of elements in the tuples extends to a bijection $\sigma_{\bar{g},\bar{q}}$ such that:

1. $tp_M(\bar{z}^{\nu}_{\bar{g}}, \bar{z}^{\nu}) = tp_M(\sigma_{\bar{g},\bar{q}}(\bar{z}_{\bar{g}^{\nu}, \bar{z}^{\nu}}))$;
2. $\sigma_{\bar{g},\bar{q}}$ fixes $\bar{z}$
3. $\sigma_{\bar{g},\bar{q}}$ resepcts the $1^{\nu}, p, 1^{\iota}$ parts and the handles.

Homestretch of the proof

- Considering all $\bar{I}$ and $\langle \bar{I}_g \mid g \in G_{k,p} \rangle$ in $\langle L \rangle$ as a saturated moel of $Th(\langle L \rangle)$, by QE, we have that $\langle \bar{I}_g \rangle$ is both $O_{k,p}$ and $G_{k,p}$ indiscernible over $\bar{I}$ in $\langle L \rangle$.

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$$G \models \theta(\bar{z}^\frown \bar{I}; \overline{(\bar{z}^\frown \bar{I})_{\bar{g}}) \wedge \neg\theta(\bar{z}^\frown \bar{I}; \overline{(\bar{z}^\frown \bar{I})_{\bar{q}}})}$$

Homestretch of the proof

- Considering all $\bar{l}$ and $\langle \bar{l}_g \mid g \in G_{k,p} \rangle$ in $\langle L \rangle$ as a saturated moel of $Th(\langle L \rangle)$, by QE, we have that $\langle \bar{l}_g \rangle$ is both $O_{k,p}$ and $G_{k,p}$ indiscernible over $\bar{l}$ in $\langle L \rangle$.
- By stability, $\langle L \rangle$ is k-dependent, and so $\langle \bar{l}_g \rangle$ will be $O_{k,p}$ indiscernible over $\bar{l}$
- For $\bar{g}, \bar{q} \in G_{k,p}$ such that $g_i, q_i > \lambda_i$ and $g_i, q_i \in P_i$, and $G_{k,p} \models R(\bar{g}) \wedge \neg R(\bar{q})$, by the choice of $\bar{z}^\frown \bar{l}$,

$$G \models \theta(\bar{z}^\frown \bar{l}; \overline{(\bar{z}^\frown \bar{l})_{\bar{g}}}) \wedge \neg\theta(\bar{z}^\frown \bar{l}; \overline{(\bar{z}^\frown \bar{l})_{\bar{q}}})$$

- Since we have an automorphism sending $\bar{g}$ to $\bar{q}$, we have a contradiction.

Problems to consider

- Are there pseudofinite strictly k-dependent groups for $k > 2$
- Are there $\aleph_0$ categorical strictly k-dependent groups for $k > 2$
- Are there strictly k-dependent fields for $k \geq 2$