Introduction to the Surreal Numbers

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- Operations, Limits, and Gaps

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- Case 3: symmetric to Case 2
- Case 4: $L, R \neq \emptyset$. Take α as before. Break into two cases

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- For any other $L < e < R$, $e(\beta) = d(\beta)$ for all $\beta \in \alpha$ by lexicographical ordering, so d is an initial segment of e, and also by lexicographical ordering d' is an inital segment of e'.

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- A similar argument is run if R has elements with initial segment d

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- A similar argument works for $d \in L$.

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Cofinality

- (F, G) is cofinal in (L, R) if for all $a \in L$ there is an $a' \in F$ such that $a \leq b$ and for all $b \in R$ there is a $b' \in G$ such that $b \leq a$.

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- One immediate consequence of these theorems: for any $a \in$, if $a_L = \{b \mid b < a \wedge b \subset a\}$ and $a_R = \{b \mid a < b \wedge b \subset a\}$, then $a = LR$.

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- Armed with these results we can begin to define algebraic operations.

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- So is an ordered abelian group.
- (Uniformity) For any representations $a = LR, b = FG$,
  $a + b = l + b, a + f\, r + b, a + g$. This follows by inverse cofinality where L is cofinal in a_L, and so on for the other sets.

Multiplication

- Define $ab =$
  $a_L b + ab_L - a_L b_L, a_R b + ab_R - a_R b_R a_L b + ab_R - a_L b_R, a_R b + ab_L -$

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- By induction on the natural sum of the lengths of each of the factors we find that ab is always defined and for $a > b$ and $c > d$ that $ac - bc > ad - bd$.

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- By induction on the natural sum of the lengths of each of the factors we find that ab is always defined and for $a > b$ and $c > d$ that $ac - bc > ad - bd$.(Specifically, let P(a,b,c,d) denote the inequality $ac - bc > ad - bd$, find that P is transitive, and induct on the proper initial segments)

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- Suppose $a > 0, b > 0$, then $P(a, 0, b, 0)$ follows, ie $ab > 0$.

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- In turn, is RCF, and inverses can be found using traditional formal power series.

Defining exp

- Gonshor uniformly defined an exponential operation on via

$$\exp(a) = \{0, (\exp a_L)[a - a_L]_n, (\exp a_R)[a - a_R]_{2n+1}\}$$

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$$\ln(\omega^b) := \ln(\omega^{b_L}) + n, \ln(\omega^{b_R}) - \omega^{\frac{b_R - b}{n}} \ln(\omega^{b_R}) - n, \ln(\omega^{b_L}) + \omega^{\frac{b - b_l}{n}}$$

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Types of Gaps and a topology

- Gaps come in two types with the following normal forms:

$$(I) \sum_{i \in} \omega^{y_i} r_i$$

$$(II) \sum_{i \in \alpha} \omega^{y_i} r_i \oplus (\pm \omega^{\Theta})$$

where Θ is a gap whose right class contains all y_i,
$n \oplus g = n + g_L n + g_R$, and $\omega^{\Theta} = 0, \omega^l a \omega^r b$ with $a, b \in_{>0}$ and
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- (Off,∞) is an open interval by $(\infty,)$ is not.