Introduction to Analysis of Surreal-valued Genetic Functions

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Outline

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- Recall, a partizan game G is position-closed if for all $X \in L_G$ and $Y \in R_G$, $L_X, R_Y \subseteq L_G \cup R_G$;

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- Numbers can be understood as unique minimal realization of cuts which correspond functions $\alpha \to 2$, and $a <_s b$ if for some $\beta \in \alpha$, $b \restriction \beta = a$.
- We want to use the simplicity hierarchy to define surreal-valued functions

Two Key Results

Theorem (Gonshor Inverse Cofinality Theorem)

For all $a \in \mathrm{No}$*, if* $a = F|G$*, then* (F, G) *is cofinal in* (L_a, R_a)*.*

Theorem (Conway's Simplicity Theorem)

Let $L, R \subset \mathrm{No}$ *such that* $L < R$ *and* $L \cup R \neq \mathrm{No}$*. Let* $I = \{y \in \mathrm{No} \colon L < y < R\}$*. Then* I *is a non-empty convex class for which there exists a unique* $x \in I$ *such that* $\iota(x) < \iota(y)$ *for all* $y \in I \backslash \{x\}$*.*

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- We let $\mathscr{S}(F|G) = \{x \in \mathrm{No} \colon F < x < G\}$
- We denote the simplest surreal number in $\mathscr{S}(F|G)$ by $F|G$.

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- $\mathscr{C}(X) = \{(L, R) \colon L \cup R = X \wedge L < R\}$
- Let $\chi(X) = X \cup \mathscr{C}(X)$ denote the Cuesta Dutari completion of X, ordered by:
  1. if $x, y \in X$, then x and y are ordered as in X;
  2. if $x \in X$ and $y = (L, R) \in \mathscr{C}(X)$, then $x < y$ if $x \in L$ and $y < x$ if $x \in R$;
  3. if $x = (L, R)$ and $y = (F, G)$ in $\mathscr{C}(X)$ such that $L \neq F$, then $x < y$ if $L \subsetneq F$, o.w. $y < x$.

Cuesta-Dutari Cuts (In General)

- It is a routine proof by cases to verify that $\chi(X)$ is an ordered Class.
- $\chi(X)$ contains its infimum and supremum, at Cuesta-Dutari cuts $(X, \emptyset)$ and $(\emptyset, X)$.
- For ordered Class X
  1. For all $x < y$ in X, there is $c \in \mathscr{C}(X)$ such that $x < c < y$;
  2. For all $c < d \in \mathscr{C}(X)$ there is $x \in X$ such that $c < x < d$.

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- Denote by $O_\alpha = X_\alpha$. Set $N_\alpha = X_{\alpha+1} \setminus X_\alpha$, and finally set $M_\alpha = X_{\alpha+1} = O_\alpha \cup N_\alpha$.
- $$\mathrm{No} = \bigcup_{\alpha\in\mathrm{On}} O_\alpha = \bigcup_{\alpha\in\mathrm{On}} M_\alpha.$$
- $(N_\alpha)_\alpha$ partitions No, in fact $N_\alpha = Lev_{\mathrm{No}(\alpha)}$.

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  1. $\sum_{\mathrm{On}} \omega^{y_i} r_i$, where (y_i) is a descending sequence of surreals, and r_i are non-zero reals.
  2. $\sum_{\alpha} \omega^{y_i} r_i \oplus (\pm\omega^{\Theta})$, with (y_i) and (r_i) as above, and Θ a gap whose Right option Class contains all of the y_i, and $\oplus$ denotes the sum of a number a and a gap g, i.e. $a \oplus g = \left\{a + g^{\mathfrak{L}}\right\} \Big| \left\{a + g^{\mathfrak{R}}\right\}$, and $\omega^{\Theta} = \left\{0, a\omega^{l}\right\} \Big| \left\{b\omega^{r}\right\}$, and $a, b \in_{>0}$, $l \in \mathfrak{L}_{\Theta}$, and $r \in \mathfrak{R}_{\Theta}$.

Immediate Results

- For all $A \subseteq \mathrm{No}$, we define the classes

$$\mathfrak{L}_{r.\sup A} = \{y \in \mathrm{No} \colon \exists a \in A (y \leq a)\}$$

$$\mathfrak{R}_{r.\inf A} = \{y \in \mathrm{No} \colon \exists a \in A (a \leq y)\}$$

- For all $a \in \mathrm{No}$, and every $(F|G) \in \mathfrak{E}^*$ such that $F|G = a$, $(\mathfrak{L}_a, \mathfrak{R}_a)$ is cofinal in (F, G).

Immediate Results

Theorem

Let $\langle (F_\alpha | G_\alpha) \rangle_{\alpha \in \mathrm{On}}$ *be an* On *length sequence of Conway cuts in* $\mathfrak{E}$ *such that for all* $\alpha, \beta \in \mathrm{On}$:

1. $F_\alpha < G_\beta$;
2. $F_\alpha \subset F_\beta$ *and* $G_\alpha \subset G_\beta$ *for all* $\beta \ni \alpha$.

Then

1. $(\bigcup_{\alpha \in \mathrm{On}} F_\alpha | \bigcup_{\alpha \in \mathrm{On}} G_\alpha)$ *is realized by* $\bigcap_{\alpha \in \mathrm{On}} \mathscr{S}(F_\alpha | G_\alpha)$;
2. $\bigcap_{\mathrm{On}} \mathscr{S}(F_\alpha | G_\alpha)$ *is empty if and only if* $\bigcup F_\alpha | \bigcup G_\alpha$ *is a gap.*

Dedekind completion (what gaps to worry about)

- Let $\mathfrak{X}$ be a space of Conway cuts. Let $[\mathfrak{X}]$ denote the space of realizations of cuts with minimal set-theoretic rank up to On.
- Suppose $A, B \subset \text{No}$ such that $A \subset B$. Then $\mathfrak{L}_{r.\sup A} \subset \mathfrak{L}_{r.\sup B}$ and $\mathfrak{R}_{r.\inf B} \subset \mathfrak{R}_{r.\inf A}$
- Define the **Cuesta-Dutari operator** $(-)^{\mathfrak{N}} : \mathfrak{E}^* \to [\mathfrak{E}^* \backslash \mathfrak{E}]$ by
$$(F|G) \mapsto \mathfrak{L}_{r.\sup F}|\mathfrak{R}_{r.\inf G}.$$
- Define the **Dedekind operator** $(-)^{\mathfrak{D}} : \mathfrak{E}^* \to [\mathfrak{E}^* \backslash \mathfrak{E}]$ by
$$(F|G) \mapsto \mathfrak{L}_{(F|G)^{\mathfrak{N}}}|\mathfrak{R}_{(F|G)^{\mathfrak{N}}}$$
- An **analytic gap** $\mathfrak{g}$ is represented by any $(F|G) \in \mathfrak{E}^*$ such that there does not exist an $a \in \text{No}$ so that $(F|G)^{\mathfrak{D}} = a$.

The Big Idea

We want to find a recursive scheme to define options sets such that they dominate the Dedekind representation.

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- Specifically, given some $\mathbb{G} \trianglelefteq PG$, we want to study the endomorphisms of $\mathbb{G}$ that are recursively definable with respect to partizan games whose option sets are recursively constructed on $\mathbb{G}$ with respect to games and their canonical realizations.

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- Issues arise when we cannot ensure that $=$ - equivalence relation is well-defined in $\mathtt{NBG}$. We want uniformity relative to $\mathbb{G}$

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- For any $A \subseteq \mathrm{No}$ that is convex and non-empty, there is a unique simplest element $a \in A$.
- We have $L|R = F|G$ if and only if for all $x^L \in L, x^R \in R, x^F \in F, x^G \in G$,

$$x^L < F|G < x^R$$

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$$ab = \{a^L b + ab^L - a^L b^L, a^R b + ab^R - a^R b^R\} |$$
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- elements $f^o \in L \cup R$ are options of f, here defined on codomain No and domain classes $A \times B$ with

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This is recursive as $f(x^o)$ is already uniquely defined for every $x^o \in L_x \cup R_x$ as the simplest realization of the cut.

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- Fornasiero proposes the strict condition that $f = L|R$ is uniform if and only if for all x,y,z such that $y < x < z$,

$$f^L(x, y, z, f(y), f(z)) < f(x) < f^R(x, y, z, f(y), f(z))$$

and for any representation of x, then

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- One can extend this to arbitrary n-arity functions by extending the partial order $\leq_s$ on No to a well-founded partial order on No^n.

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$$\begin{aligned} 0 &= \iota(a + (-a)) \\ &= \end{aligned}$$

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- Importantly, for my talk they proposed the following inductive construction for genetic functions

Rubinstein-Salzedo and Swaminathan

- Consider the Ring (capitalized because **No** is a proper class) K obtained by adjoining to **No** the following collection of symbols: $\{g(a), g(b) : g \in S \cup \{f\}\}$, where a, b are indeterminates (notice that the symbols $f(a), f(b)$ are allowed, but no other symbols involving f are allowed). We then obtain a class of symbols $S(a,b) = \{c_1 h(c_2 x + c_3) + c_4 : c_1, c_2, c_3, c_4 \in K, h \in S\}$ (now notice that $h \in S$, so we cannot take h to be f in this part of the construction). Next, consider the Ring $R(a,b)$ generated over **No** by adjoining the elements of $S(a,b)$, and let $L_f, R_f \subset R(a,b)$ be proper subsets. Fix $x \in$ **No**, and suppose that $f(y)$ has already been defined for all $y \in L_x \cup R_x$. Also let $x^L \in L_x, x^R \in R_x$. Then replace a with x^L and b with x^R in $R(a,b)$ and consider the resulting sets of functions $L_f(x^L, x^R), R_f(x^L, x^R)$ from **No** $\to$ **No** (these sets are obtained from L_f, R_f by substitution). Now, if for all $x^L, x^{L'} \in L_x$, $x^R, x^{R'} \in R_x$, $f^L \in L_f(x^L, x^R)$, $f^R \in R_f(x^{L'}, x^{R'})$ we have $f^L(x) < f^R(x)$, we let $f(x)$ be given by the expression

$$\left\{ \bigcup_{x^L \in L_x, x^R \in R_x} \{f^L(x) : f^L \in L_f(x^L, x^R)\} \,\middle|\, \bigcup_{x^L \in L_x, x^R \in R_x} \{f^R(x) : f^R \in R_f(x^L, x^R)\} \right\}.$$

  In this case, f is genetic. The elements of L_f are called left options of f and denoted f^L, and the elements of R_f are called right options of f and denoted f^R.

Issues with Rubinstein-Salzedo and Swaminathan

1. Unclear how we form important otherwise genetic functions like exp
2. Uniformity is not guaranteed (this admits ι)

Amendments to Rubinstein-Salzedo and Swaminathan

Let v, w denote indeterminates, and let $f : \text{No} \to \text{No}$ be a function symbol, and suppose S is a set of genetic functions that have already been defined. Then

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2. We obtain a Class

$$S(v, w) = \{c_1 + c_2 h(c_3 x + c_4) \colon c_1, c_2, c_3, c_4 \in K, h \in S\}.$$

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3. We then form Ring $R(v, w) := \text{No}[S(v, w)]_{P_S}$, where P_S is the cone of strictly positive polynomials with function from S.

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- Fix an $x \in \text{No}$, and suppose $f(y)$ has already been defined for all $y \in L_x \cup R_x$, substitute v with x^L and w with x^R in $R(v, w)$
- Choose sets $L_f(x^L, x^R)$, $R_f(x^L, x^R)$ from $R(x^L, x^R)$ such that the order condition holds, i.e. for all $x^L, x^{L'} \in L_x$ and $x^R, x^{R'} \in R_x$, and $f^L \in L_f(x^L, x^R)$ and $f^R \in R_f(x^{L'}, x^{R'})$ we have $f^L(x) < f^R(x)$, and

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- and also the cofinality condition holds,

$$\forall x, y, z \in \mathrm{No}((y < x < z) \rightarrow$$

$$L_f(y, z)[x] < f(x) < R_f(y, z)[x].$$

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- Once f is defined over No, we prove that the cofinality condition holds, via (double) induction with respect to the natural sum of the lengths of the arguments and generation.

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Finally, set

$$f(x) := \{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^L(x) \colon f^L \in L_f(x^L, x^R)\}\} |$$

$$\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^R(x) \colon f^R \in R_f(x^L, x^R)\}\}$$

Proof of Uniformity

Theorem

Suppose f is defined with respect to option term sets $L_f, R_f \subset R(v, w)$ as above. Then f is a surreal-valued genetic function if and only if f has the uniformity property.

Proof.

In the reverse direction, if f is a recursively defined surreal-valued function invariant under representation defined by $L_f, R_f \subset R(v, w)$, then necessarily the order property must be satisfied, as otherwise there would be some Left option and Right option and $b < a < c$ such that $f^L(b, c; a) \geq f^R(b, c; a)$, whence the game value defined will not be numeric, or $f^L(b, c; a) \geq f(a)$ or $f^R(b, c; a) \leq f(a)$. □

Proof of Uniformity

Proof.

In the forward direction, suppose that f is a surreal-valued genetic function. Then f satisfies the order condition and the cofinality condition. We wish to show that for all $a \in \mathrm{No}$ and all representations $(F|G)$ of a in $\mathfrak{E}^*$, we have

$$
\begin{aligned}
f(a) &= \left\{ \bigcup_{F,G} \{ f^L(x) \colon f^L \in L_f(b,c) \} \right\} \Bigg| \\
&\quad \left\{ \bigcup_{F,G} \{ f^R(x) \colon f^R \in R_f(b,c) \} \right\} \\
&= L_f(L_a, R_a; a) | R_f(L_a, R_a; a)
\end{aligned}
$$

But this follows by the global cofinality condition. □

Finally, the Composition rule

If $g, h \in \mathfrak{G}$, set $f = g \circ h$. If for all $x \in \text{No}$, $g(x) := L_g(x^L, x^R) | R_g(x^L, x^R)$ and similarly, $h(x)$ is given in terms of sets of genetic Left and Right formula given with respect to the Left and Right options of x, then

$$f(x) := \left\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \bigcup_{\substack{h^L \in L_h(x^L, x^R) \\ h^R \in R_h(x^L, x^R)}} \{g^L(h(x)) \colon g^L \in L_g(h^L(x), h^R(x))\} \right\} \Bigg|$$

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Questions