Ordinal-valued functions identifying models of algebraic theories

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The goal of this paper is to collect and prove necessary and sufficient conditions for a a specific class function $S : \textsc{On} \to \textsc{Sets}$ to satisfy

$$S(\alpha) \models T$$

for various algebraic theories T. Namely, we consider the class function S defined by the mapping $\alpha \mapsto (\bigcup_{\beta \in \alpha} {}^{\beta}2, \sqsubset)$, which sends an ordinal to a binary tree of height α, and consider the necessary and sufficient conditions an ordinal must satisfy to specify a given algebraic theory. Specifically, the algebraic theories we will investigate will be over signatures with $+, \cdot, v$ derived from the surreal numbers.

1 Initial Correction and some necessary $\leqslant_s$ results

Remark. *While it is quite embarrassing that the results below stem from my simultaneously misstating and misattributing an interesting result of Conway, an embarrasment compounded by the fact that some of the work I did recently for Dave involved the characteristic p examples described below, nonetheless, these results are novel and interesting in and of themselves. Moreover, the result which was misattributed to Conway is a consequence of Chapter 6 in Gonshor, where ϵ numbers are identified with real closed fields.*

The initial misstatement was that I attributed to Conway a result that held that subtrees of No *of height the epsilon numbers were real closed fields. This is simply not the case, as an immediate counter example would have the full binary subtree of height $\omega_0 = \omega(0)$ is a model of* RCF*, and $\omega(0) \ll \epsilon(0)$.*

The actual ϵ "result" of Conway occurs in Chapter 6 of [2]. In this chapter, Conway begins by showing how one can turn On *into a field of characteristic 2 by means of an inductive construction identifying the* minimal excludent */cite{ONAG, Onp}. This construction is based on Nim-arithmetic that Conway explores elsewhere in [2]. The ordinal results that Conway provides in this chapter characterise when a given ordinal is a group, a ring, a field, etc.*

Specifically, mex$\{S\}$ indicates the least ordinal not in a set S, with the members of S being called excludents. This gives rise to

$$\alpha + \beta = mex\{\alpha' + \beta, \alpha + \beta'\}$$

$$[-\alpha] = mex\{[-\alpha']\}$$

$$\alpha\beta = mex\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$$

$$\beta = \alpha^{-1} = mex\left\{0, \frac{1 + [-\alpha' - \alpha]\beta'}{\alpha'}\right\}$$

$$\beta = \sqrt{\alpha} = mex\left\{\sqrt{\alpha'}, \frac{\alpha'\alpha^* + \alpha}{\alpha' + \alpha^*}\right\}$$

with α', α^* indicating distinct ordinals less than α.

With square brackets used to indicate ordinal arithmetic operations distinct from the field operations under consideration, Δ to indicate an ordinal whose arithmetic relation to earlier ordinals is being considered, and $\delta \in \Delta$, Conway proves the following:

Fact 1.

1. *If Δ is a not a group under addition, then $\Delta = \alpha + \beta$, where (α, β) is a any lexicographically earlier pair of numbers in Δ whose sum is not in Δ.*

2. *If Δ is a group, then $[\Delta\alpha] + \beta = [\Delta\alpha + \beta]$ for all $\alpha \in$ ON and $\beta \in \Delta$.*

3. *If Δ is a group but not a ring, then $\Delta = \alpha\beta$, where (α, β) is any lexicographically earliest pair of numbers in Δ whose product is not in Δ.*

4. *If Δ is a ring, and $\Gamma \leqslant \Delta$ is an additive subgroup all of whose non-zero elements have multiplicative inverses in Δ, then $\Delta\gamma = [\Delta\gamma]$ for all $\gamma \in \Gamma$.*

5. *If Δ is a ring, but not a field, then Δ is the inverse of the earliest non-zero α in Δ which has no inverse in Δ.*

6. *If the assumptions in Fact 1.5 hold, then $\Delta^n\gamma_n + \Delta^{n-1}\gamma_{n-1} + \cdots + \Delta\gamma_1 + \delta = [\Delta(\Gamma^{n-1}\gamma_n + \cdots + \gamma_1) + \delta]$ for all $n \in \omega$ and all $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$, and $\delta \in \Delta$.*

7. *If Δ is a field but not algebraically closed, then Δ is a root of the lexicographically earliest polynomial having no root in Δ (by examining high degree coefficients first).*

8. *Same assumptions as Fact 1.7, for all $n < N$ and $\delta_0, \ldots, \delta_n \in \Delta$, then $\Delta^n\delta_n + \cdots + \delta_0 = [\Delta^n\delta_n + \cdots + \delta_0]$.*

9. *If Δ is an algebraically closed field, then Δ is transcendental oer Δ, adn we have \$$\Delta n\delta_n + \ldots + \delta_0 = [\Delta^n\delta_n + \ldots + \delta_0]$ for all $n \in \omega$ and $\delta_0, \ldots, \delta_n \in \Delta$.*

10. *If Δ is a group, then $[\Delta 2]$ is also a group. Moreover, the ordinals that are groups are precisely the 2-powers $[2^{\alpha}]$, with each ordinal decomposing uniquely into a finite sum of descending 2-powers, with the sums agreeing in both senses considered (as ordinals, and as algebraic objects).*

11. *The finite fields are Fermat 2-powers.*

12. *The first few infinite fields are*

$$\omega, [\omega^3], [\omega^9], \ldots,$$

then

$$[\omega^{\omega}], [\omega^{\omega 5}], [\omega^{\omega 25}], \ldots,$$

and in general for p the (k+1)st odd prime, with α_p the least number in $[\omega^{\omega^k}]$ with no pth root in $[\omega^{\omega^k}]$, the sequence

$$\alpha_p, [\omega^{\omega^k}], [\omega^{\omega^k p}], [\omega^{\omega^k p^2}], \ldots$$

describes the next set of fields.

What Conway suggested, and diMuro showed, is that $\epsilon(0)$ as the ordinal of the first cubic extension $\sqrt[3]{t}$. Moreover, diMuro generalizes Conway's minimal excludent construction to general characteristic p cases. In both cases, the problem of identifying the next transcendental is raised and phrased as follows:

Question 1. *How does one express in terms of ordinal arithmetic the least ordinal greater than $\omega^{\omega^{\omega}}$ which is transcendental over previous ordinals. In particular, decide how this relates in realtion to the ordinal ω_0 and the least α*

These arguments will be revisited in the subsection on fields below. Specifically, many of the results to come can be traced back to work done by Philip Ehrlich [3] and his works on /Number Systems with Simplicity Hierarchies: A Generalization of Conway's Theory of Surreal Numbers/. Specifically, Ehrlich shows that in addition to be the *unique* **homogeneous universal ordered field**, the surreal numbers structure as a lexicographically ordered binary tree have an algebraico-tree-theoretic structure such that the surreals are teh unique complete s-hierarchical (ordered) group, field, vector space, and general s-hieararchical ordered structure, and that there is one and only one s-hieararchical map from an s-hierarchical ordered structure into No.

These maps are specifically identified as a initial subtrees T of No, which will properly be subtrees of No. These maps in turn allow one to characterize No up to isomorphism as the unique complete, nonextensible and universal s-hierarchical group/field/vectors space. Ehrlich also considers the spectrum of *initial* substructures of No, which includes every real-closed ordered field being isomorphic to an initial subfield of No.

Definition. *A ordered binary tree (given by $\mathcal{L} = \{T, <, <_s\}$) is { lexicographically ordered} whenever for all $x, y \in T$, where $x < y$, x is incomparable with y if and only if there is $z <_s x, y$ such that $x < z < y$.*

Whenever T is an ordered tree, then $x <_s y$ is read as "x is simpler *than y". An element x of a non-empty subclass I of T is the* ***simplest member*** *if for all $y \in I \backslash \{x\}$, $x <_s y$. Following [3],* ***s-hierarchies*** *arise as follows:*

1. *$\langle T, +, <, <_s, 0\rangle$ is an* ***s-hiearchical*** *order group if i. $\langle T, +, <, 0\rangle$ is an ordered abelian group; ii. $\langle T, +, <_s\rangle$ is a lexicographically ordered binary tree; iii. for all $x, y \in A$, $x + y = \{x^L + y, x + y^L\} \mid \{x^R + y, x + y^R\}$*

2. *$\langle T, +, \cdot, <, <_s, 0, 1\rangle$ is an* ***s-hierarchical ordered field*** *if: i. $\langle T, +, \cdot, <, 0, 1\rangle$ is an ordered field; ii. $\langle T, +, <, <_s, 0\rangle$ is an s-hierarchical ordered group; iii. for all $x, y \in T$,*

   $$xy = \{x^L y - xy^R - x^L y^L, x^R y + xy^R - x^R y^R\} \mid \{x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}$$

3. *$\langle T, +, \cdot, <, <_s\rangle$ is an *s-hierarchical ordered vector space over K if: i. K is an s-hierarchical ordered field; ii. T is an ordered K-vector space in which for all $x \in K$ and $y \in T$, xy is defined by the same inductive game definition above.*

A subgroup (subfield, subspace) A of an s-hierarchical group (s-hierarchical ordered field, s-hierarchical ordered vector space) T is ***initial*** *if A is an initial subtree of T. We say A is a* ***maximal initial*** *subtree of T if for all other initial subtrees S of T such that $ht(S) \leqslant ht(A)$, then $S \subset T$. Specifically, we identify all maximal initial binary trees with the subsets $\{x \in \text{No} \mid lh(x) \leqslant \alpha\}$, where $\alpha \in \text{On}$.*

A binary tree T is ***full*** *if every element has exactly two successors, and every chain of infinite limit length $<$ On, has an immediate successor.*

A lexicographically ordered binary tree T is ***complete*** *if for all subsets $L, R \subset T$ such that $L < R$, there is a $y \in T$ such that $y = \{L\} \mid \{R\}$.*

A mapping $f : A \to B$ between two lexicographically ordered binary trees A and B is ***s-hierarchical*** *if for all $x \in A$,*

$$f(x) := \{f(x^L)\} \mid \{f(x^R)\}$$

An s-hierarchical group (field, vector space) U is ***universal*** *if there is an s-hierarhical embedding $f : A \to U$ for each s-hierarhical group (resp. field, vector space) A.*

An s-hiearchical ordered group (s-hierarchical ordered field; s-hierarchical ordered vector space) M is ***maximal*** *or* ***non-extensible*** *if there is no s-hierarchical structure that properly contains M as an initial substructure.*

Remark. *The last two definitions are made in part to correct Conway's misleading description of No as an "universally embedding" structure. Properly understood, the surreal numbers are universally extending structure in the sense of Ehrlich, et al. This is to correct an error of Dales and Woodin who mistakenly asserted in the literature that the surreal numbers are up to isomorphism the universe universal ordred field.*

Ehrlich [3] correct these assertion and provides the following result of interest:

1. *Every divisible ordered abelian group is isomorphic to an initial subgroup of* No*;*

2. *Every real-closed ordered field is isomorphic to an initial subfield of* No.

3. *every non-trivial s-hierarchical ordered group (field) A contains a cofinal, canonical subsemigroup (subsemiring)* $\mathrm{On}(A)$*, the ordinal part of A, contained in a discrete, canonical subgroup* $\mathrm{Oz}(A)$*, the omnific integer part of A, in each for each $x \in A$, there is a $z \in \mathrm{Oz}(A)$, and e the least positive element of $\mathrm{Oz}(A)$ such that $z \leqslant x < z + e$,*

The following two theorems are important for characterizing the initial subtreeswhich we are interested in identifying: [3]

Theorem 1. *Given an ordered tree $A = \langle A, <, <_s \rangle$, the following are equivalent:*

1. *$\langle A, <, <_s \rangle$ is a lexicographically ordered binary tree;*

2. *$\langle A, <, <_s \rangle$ is uniquely isomorphic to an initial subtree of $\langle \mathbf{B}, <_{lex}, <_{\mathbf{B}} \rangle$, with $\mathbf{B} = \bigcup_{\beta \in \mathrm{On}} {}^{\beta}\{-, +\}$ the canonical binary tree;*

3. *every nonempty convex subclass of A contains a simplest memeber wrt $<_s$, and for all $x, y \in A$, if $x <_s y$ then $x^L < y < x^R$.*

Theorem 2. *For a lexicographically ordered binary tree $\langle T, <, <_s \rangle$, the following are equivalent:*

1. *$\langle T, <_s \rangle$ is a full binary tree;*

2. *$\langle T, <, <_s \rangle$ is complete;*

3. *the intersection of every nested sequence I_α of non-empty convex subclasses of $\langle T, <, <_s \rangle$ is nonempty (and thus would have a simplest member by the result above).*

We also note that $\omega(x)$ are the simplest positive elements of their respective Archimedean class, with proofs of this available in [1, 3, 4].

2 Sign Sequence Primer

The following results are a summary of Gonshor Chapter 5 [4], as well as some new results of Kuhlmann and Matusinski [6]. While each author has their own preferred notation for concatenation and representing the sign sequence, we have opted to use notation keeping in line with work found in Kunen [7], Jech [5], and other more set-theoretically inclined authors [8].

Definition. *Following Gonshor, a surreal number a can be regarded as a function from some ordinal α to a set of cardinality 2. These functions are so that for two surreal numbers a, b, we may* ***concatenate*** *them to form a third number,*

$a \frown b$. The concatenation operation respects standard results on ordinal length, i.e.

$$lh(a \frown b) = lh(a) \oplus lh(b)$$

as can be verified by an induction argument on the lengths of numbers.

Whenever necessary, we will use $\oplus$ to indicate ordinal addition, and $\otimes$ to indicate ordinal multiplication. Otherwise, from context $\alpha\beta$ and other string concatenations of Greek letters will indicate ordinal multiplication.

Notation. *We will denote by (a) the sign sequence of a, and write out the sign sequence as a sequence of ordered pairs $(\langle \alpha_\mu, \beta_\mu \rangle : \mu \in \lambda)$ for some $\lambda \in \mathrm{On}$. Note, if $\alpha_\mu = 0$, then $\mu = 0$ or $\mu \in Lim(\mathrm{On})$, and if $\beta_\mu = 0$, then μ is the maximum element in λ, i.e. the sequence terminates at μ.*

Definition. *Given $a \in \mathrm{No}$, let a^+ denote the total number of $+$ appearing in the sign sequence of a, so*

$$a^+ = \sum_\mu \alpha_\mu$$

as an ordinal sum.

Given $a \in \mathrm{No}_{>0}$, define $a^\flat$ to be the surreal number attained by omitting the first $+$ sign.

Given $a \in \mathrm{No}_{<0}$, define $a^\sharp$ to be the surreal number attained by omitting the first $-$ sign.

Given a surreal in normal form $a = \sum_{i \in \lambda} \omega^{a_i} r_i$, the ***reduced sequence*** *$(a_i^o | i \in \lambda)$ is attained by omitting "-" in the following sign sequences:*

- *given $\gamma \in \mathrm{On}$, if $a_i(\gamma) = -$ and there exists $j < i$ such that \$$a_j(\delta)$=$a_i(\delta)$ for all $\delta \leqslant \gamma$, then omit the δ^{th} "-";*
- *if i is a successor, $a_{i-1} \frown - \sqsubset a_i$ and if r_{i-1} is not a dyadic rational, then omit the - after a_{i-1} in a_i.*

The following theorems provide a concise overview of the sign sequence lemma, as well as the sign sequence of generalized epsilon numbers.

Theorem 3. *Given $a = (\langle \alpha_i, \beta_i \rangle)_{i \in \lambda}$, for any $\mu \in \lambda$, we set*

$$\gamma_\mu := \sum_{\lambda \leqslant \mu} \alpha_\lambda$$

Then ω^a has the sign sequence

$$\langle \omega^{\gamma_0}, \omega^{\gamma_0 + 1} \beta \rangle \frown (\langle \omega^{\gamma_i}, \omega^{\gamma_1 + 1} \beta_i \rangle)_{0 < i < \mu}$$

Theorem 4. *Given a positive real r with sign sequence $(\langle \rho_i, \sigma_i \rangle)$, the sign sequence of $\omega^a r$ is*

$$(\omega^a) \frown \langle \omega^{a^+} \rho_0^\flat, \omega^{a^+} \sigma_0 \rangle \frown (\langle \omega^{a^+} \rho_i, \omega^{a^+} \sigma_i \rangle)$$

with $\omega^{a^+} \rho$ and $\omega^{a^+} \sigma$ being the standard ordinal multiplication (with absorption).

If r is a negative real, we reverse the signs.

Theorem 5. *Given* $a = \sum_{i<\lambda} \omega^{a_i} r_i$,

$$(a) = \frown_{i<\lambda} (\omega^{a_i^o} r_i)$$

The following theorem is a combination of theorems 9.5 and 9.6 in [4]

Theorem 6. *1.* $a = (\langle \alpha_i, \beta_i \rangle)$ *is an epsilon number if and only if* $\alpha_0 \neq 0$, *all* $\alpha_\mu \neq 0$ *are ordinary epsilon numbers such that* $\alpha_\mu > lub\{\alpha_\lambda \mid \lambda < \mu\}$, *and* β_μ *is a multiple of* $\omega^{\alpha_\mu \omega}$ *for all* $\alpha_\mu \neq 0$, *and a multiple of* $\omega^{\gamma_\mu \omega}$ *where* $\delta_\mu = \sum_{\lambda<\mu} \alpha_\mu$ *for* $\alpha_\mu = 0$.

2. Let $\gamma_\mu = \sum_{\lambda \leqslant \mu} \alpha_\lambda$. *Then the* μ^{th} *block of* $+$ *in* $\varepsilon(a)$ *consists of* e_{γ_μ} *+'s and the* μ^{th} *block of -'s will consist of* $(\varepsilon_{\gamma_\mu})^\omega \beta_\mu$ *-'s.*

3 Additional Sign sequence results

Finally, the following results from Chapter 6 of [4] are vital for proving the results we need on binary trees with $\oplus, \otimes$ indicating ordinal addition and multiplication, and $|x|$ indicating the cardinality of x. Supposing that $lh(a) \leqslant lh(b) \leqslant lh(c)$:

Fact 2. *1.* $lh(a+b) \leqslant lh(a) \oplus lh(b)$;

2. $lh(ab) \leqslant 3^{lh(a) \oplus lh(b)}$;

3. $|lh(a^{-1})| \leqslant \aleph_0 |lh(a)|$;

4. For $a \in \mathrm{No} \backslash \mathbb{D}$, *then* $|lh(\omega(a))| = |lh(a)|$;

5. for any non-zero real r *and* a, $|lh(\omega(a)) \cdot r| = |lh(\omega(a))|$;

6. If $\omega(b)r$ *is a term in the normal from of* a, *then* $lh(\omega(b)) \leqslant lh(a)$;

7. For $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$, *then* $|\beta| \leqslant |lub_{\alpha \in \beta}[lh(a_\alpha)\omega]|$. *(This result refers to the least upper bound of ordinals on the right hand side, and cardinalities on the left hand side).*

8. For $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$, *then* $|lh(a)| \leqslant |lub_{\alpha \in \beta} lh(a_\alpha), \omega|$;

9. For $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$ *and* $lub_{\alpha \in \beta}(|\beta|, |lh(a_\alpha)|, \aleph_0) \leqslant \kappa$, *then* $|lh(a)| \leqslant \kappa$.

10. The set of surreals with lengths less than a fixed ϵ *number form a subring of surreal numbers;*

11. For $\alpha_1, \ldots, \alpha_n$ *are arbitrary surreal numbers and* $r_1, \ldots, r_n$, *are rational, then* $|lh(\sum r_i \alpha_i)| \leqslant |\max lh(\alpha_i)| \aleph_0$.

12. *An ordinal upperbound for the cardinality of κ will be the least ϵ number larger than α.*

13. *The subset of surreal numbers $\{x \mid |lh(a)| \leqslant \kappa\}$ for any fixed infinite cardinal κ will form a real closed field. Furthermore, since all operations will depend on finitely many elements of the condition $lh(a) \leqslant d$, we may strengthen this to $lh(a) < d$.*

We will explore Fact 2.12 in greater detail below.

4 Primes

A well-known result of Ernst Jacobsthal proves that the ordinals have a pseudo-unique factorization theorem. The following definitions capture the primes in $(\text{On}, \oplus, \otimes)$:

Definition. *An ordinal α is* prime *if one of the following holds: i. α is a finite prime number; ii. α is a* delta *number, i.e. $\alpha = \omega^{\omega^{\beta}}$ for some $\beta \in \text{On}$; iii. α is a successor to a γ number, that is $\alpha = \omega^{\beta} + 1$ for any $\beta \in \text{On}_{>0}$.*

Due to absorption, factorization of ordinals into primes is not unique without further conditions being applied:

Proposition 1. *An ordinal is uniquely factored into primes above provided the factorization places: i. every δ prime occurs begore every γ successor prime ii. the primes are listed in descending order for every pair of limit or finite primes (i.e if α has two delta primes α_1, α_2 with $\alpha_1 \geqslant \alpha_2$, then $\alpha = \alpha_1\alpha_2\rho$, where ρ is the remainder of α).*

Remark. *We can read off the prime factorization above by using the Cantor normal form as follows:*

1. *Write the ordinal $\alpha = \beta\gamma$ where β is the smallest power of ω in the Cantor normal form and β is a successor ordinal;*

2. *If $\beta = \omega^{\lambda}$, then expressing λ in its Cantor Normal Form gives an expansion of β as a product of its δ primes*

3. *For γ is a successor ordinal, if γ has a Cantor normal form $\gamma = \omega^{\mu_0}n_0 + \omega^{\mu_1}n_1 + \xi$, where $\xi \in \omega^{\mu_1}$, γ can further be factored into a product*

$$\gamma = (\omega^{\mu_1}n_1 + \xi_1)(\omega^{\mu_0 - \mu_1} + 1)n_0$$

as a produt of smaller ordinals.

Example 1. *Let's suppose $\alpha = \omega^{\omega^2 3}5 + \omega^{\omega 7}11 + \omega^{13}17$. We find the unique factorization of α into primes as follows:*

First, we note that $\beta = \omega^{13}$ and

$$\gamma = \omega^{\omega^2 3 - 13}5 + \omega^{\omega 7 - 13}11 + 17.$$

Since 13 is already a prime, we do not proceed to factor β any further. We proceed to factor γ as follows

5 Ordinal functions

Throughout this section, we let $S : \textsc{On} \to \textsc{Sets}$ be the function sending α to the full binary tree of height α. In other words, each branch of $S(\alpha)$ is of a length in α. We will make frequent use of the following lemma

Lemma 1. *$<_s, <$ induce ordering on the archimedean classes of* No

Proof. We note that the simplest element in each archimedean class in No is an element of the form $\omega(y)$, where $y \in \textsc{No}$. We can then induce the desired partial or linear ordering with respect to simplicity or the lexicographical ordering of No. □

Moreover, throughout this section, with S defined as above, all algebraic operations will be those defined for No. Since No and the algebraic operations are inductively defined, they will be defined inductively for all elements at a level less than the height of the binary tree.

5.1 $S(\alpha) \models$ Ab

Theorem 7. *$(S(\alpha), +_{\textsc{No}}) \models$ Ab if and only if $\alpha = \omega(\beta)$ for some $\beta \in \textsc{On}$.*

Proof. In the forward direction, suppose that $\alpha \in \textsc{On}$ is an ordinal such that $S(\alpha)$ is an abelian group.

Towards a contradiction, suppose that $\alpha \neq \omega(\beta)$ for any $\beta \in \textsc{On}$. Then α has Cantor normal form $\sum_{i \in I} \omega^{\alpha_i} n_i$ where indexing set I is either of cardinality greater than 1, or $n_0 > 1$, and (α_i) are a descending sequence of ordinals. Since $\alpha_i > \alpha_{i+1}$, we find that $\omega^{\alpha_0} n_0$ must be an element in $S(\alpha)$. But then, so will $\omega^{\alpha_0} n_0 2$. Contradiction.

Conversely, suppose that $\alpha = \omega(\beta)$. We check that $S(\alpha)$ endowed with the addition operation defined for the surreal numbers will be closed under addition. For any $a, b \in S(\alpha)$, since $lh(a), lh(b) \in \omega(\beta)$, we find that $lh(a) + lh(b) \geqslant lh(a + b)$. Let $\gamma = \max\{lh(a), lh(b)\}$. Since $\gamma \in \omega^\beta$, it follows that $\gamma 2 \in \omega^\beta$, whence $a + b \in S(\alpha)$. The abelian group axioms are satisfied by the construction of $+$, with associativity and commutativity exhaustively studied in [**?**]Gonshor, ONAG}. The existence of inverses also follows as the inverse of any surreal number is derived by changing the signs (and so the additive inverse of a surreal number of a length γ will be of length γ). □

We note that when $\beta = 0$, that the group being described consists solely of the zero element, and so the theory will vacuously be satisfied.

Remark. *As a reminder, the ordering on the Archimedean classes is identical to the lexicographical ordering on the surreals, making use of the fact that the simplest element in each archimedean class is the respective γ number of that class. Each class is closed under addition as a dense linearly ordered abelian group.*

Claim: When we cut No *off at a complete binary tree of height* $\gamma = \omega^\alpha$*, we get a* $\mathbb{Z}$ *module that is a subspace of the* $\mathbb{R}$ *vector space with basis* $\{\omega^{\pm\beta} \mid \beta \in \alpha\}$*. All non-trivial real vector spaces are odd-dimensional, but the* $\mathbb{Z}$ *submodule has restrictions on the type of coefficient that it can be multiplied against. E.g. we have something like a subspace of* $\mathbb{D}\omega^1 \oplus \mathbb{R} \oplus \mathbb{D}\omega^{-1}$ *when working at* ω^2*. This would seem to follow from* $\mathbb{D}$ *being the abelian group we're working with at height* ω*. For* ω^3 *for instance, we should have* $\omega^2 + \omega + 1$ *as an element, as well as* ω^{-1} *and* ω^{-2}*. Thus we should hav* $\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2}$ *as an element, and in general* $\mathbb{Z}$ *modules of this element, along with* $\mathbb{D}$ *modules, and at the very least* $\mathbb{R}$ *modules for all but the outer elements, provided I can show that...*

Example 2. *We may want to present the normal forms for members of* $S(\gamma)$*. When* $\gamma = \omega$*, each normal form is exactly some dyadic rational number* d*.*

When $\gamma = \omega^2$*, we have all real numbers, as well as all a normal form of*

$$\omega r_0 + r_1 + \omega^{-1} r_2$$

with the following restrictions on vectors of real numbers $\langle r_0, r_1, r_2 \rangle$*:*
if $r_0 = 0$*, then* $r_1 \in \mathbb{R}$*.*
if $r_0 \in \mathbb{D}$*, then*

Before continuing, we will need the following lemma

Lemma 2. *Let* $\lambda \in$ On*. Then* $a \in S(\omega^\lambda)$ *if and only if* $\omega(a) \in S(\omega^{\omega^\lambda})$ *.*

Proof. In the forward direction, suppose that a is a surreal number of length $\gamma \in \omega^\lambda$ with sign sequence $(\langle \alpha_\mu, \beta_\mu \rangle \mid \mu \in \xi)$, with $\gamma, \xi, \alpha_\mu, \beta_\mu \in \omega^\lambda$ for all $\mu \in \xi$.

We know that

$$\sum_{\mu \in \xi} (\alpha_\mu + \beta_\mu) = \gamma$$

and that

$$(\omega(a)) = (\langle \omega^{\gamma_\mu}, \omega^{\gamma_\mu + 1} \beta_\mu \rangle \mid \mu \in \xi)$$

so the length of $\omega(a)$ will be

$$\sum_{\mu \in \xi} (\omega^{\gamma_\mu} + \omega^{\gamma_\mu + 1} \beta_\mu).$$

If $\xi = \delta + 1$, then by left absorption

$$lh(\omega(a)) = \omega^{a^+} + \omega^{a^+ + 1} \beta_\delta.$$

If ξ is a limit ordinal, then we can strictly bound $lh(a)$ above by $\omega^{a^+ + 1} \omega^\lambda$ (since all $\beta_\mu \in \omega^\lambda$). We can further refine this by noting that $a^+ \in \omega^\lambda$, and so we are further bounded above by ω^{ω^λ}. Thus we have that $a \in S(\omega^\lambda)$ implies that $\omega(a) \in S(\omega^{\omega^\lambda})$.

Conversely, suppose that $\omega(a) \in S(\omega^{\omega^\lambda})$. Towards a contradiction, further suppose that $a \notin S(\omega^\lambda)$. Without loss of generality, let's suppose that $lh(a) = \omega^\lambda$. □

5.2 $S(\alpha) \models$ CRing

We will need the following lemma before studying multiplication:

Lemma 3. *1. $\lambda, \xi \in \alpha$;*

2. for all $\mu \in \lambda$, we have $\alpha_\mu, \beta_\mu \in \alpha$;

3. for all $i \in \xi$, $y_i \in S(\alpha)$.

Proof. First note that $\sum_{\mu\in\lambda}(\alpha_\mu + \beta_\mu) \geqslant \lambda$, since at most $\alpha_\mu = 0$ for all $\mu \in Lim(\omega^{\omega^\beta}) \cup \{0\}$, and the sequence terminates whenever $\beta_\mu = 0$. If $\lambda \geqslant \omega^{\omega^\beta}$, then we could not have a sign sequence of length less than ω^{ω^β}.

If $\xi \geqslant \omega^{\omega^\beta}$, then since the length of each summand in the normal form of a is at least size 1, we also easily derive a contradiction.

For the second claim, suppose that for μ, we had that $\alpha_\mu + \beta_\mu \geqslant \alpha$. Then immediately the length of the sequence is at least α. Contradiction.

For the third claim, if any $y_i \notin S(\alpha)$, then wlog, let $lh(y_i) = \alpha$. Since $\omega(y_i)$ is given by the sign sequence

$$(\langle \omega^{\gamma_\mu}, \omega^{\gamma_\mu+1}\beta_\mu \rangle \mid \mu \in \lambda)$$

We claim that if y_i is of length at least α, then $\omega(y_i)$ will be at least length $\omega(\alpha)$.

This claim be verified by an induction argument. If y is length zero, then $\omega(0) = 1$. Suppose this holds for all y of length α. Then for any length $\alpha + 1$, since the function is extended either by a '+' or '-' symbol, we let y be a surreal number with length α and γ many alternating pairs:

1. If $\gamma = \lambda + 1$, and $\beta_\lambda = 0$: i. for $z = y \frown +$, the ordinal $z^+ = y^+ + 1$, and so $\omega^{z^+} = \omega^{y^+}\omega \ni \omega^{y^+} + 1$, from which we conclude that $\omega(z) \geqslant \alpha + 1$. ii. for $z = y \frown -$, we have $z^+ = y^+$, and $\beta_\lambda = 1$, so
$$lh(\omega(z)) = lh(\omega(a)) + \omega^{y^+} \geqslant \alpha + 1.$$
2. If $\gamma = \lambda + 1$, and $\beta_\lambda \neq 0$: i. and ii. are similar to the arguments above.
3. If $\gamma \in Lim(\text{On})$, then for either symbol we add, we either increase z^+ by 1 or contribute another ω^{y^+} many "-" signs. Either way, we have $\omega(z) \geqslant \alpha + 1$.

Finally, suppose this is true for all $\alpha \in \lambda \in Lim(\text{On})$. Then for y such that $lh(y) = \lambda$, we must check that $lh(\omega(y)) \geqslant \lambda$. If $y = \lambda$ or $y = -\lambda$, then this will be immediate. Otherwise, let y have $\gamma > 1$ many pairs. If γ is not a limit ordinal, since the length of $y = \lambda$, by absorption either the final pair is $\langle \lambda, 0 \rangle$ or $\langle \delta, \lambda \rangle$ where $\delta \in \lambda$. Either way we immediately find that $\omega(y) \geqslant \lambda$.

At last, if γ is a limit ordinal, then we find that either $\gamma = \lambda$ and the $\alpha_\mu, \beta_\mu \in \lambda$, or either $\bigcup_{\mu\in\gamma} \alpha_\mu = \lambda$ or $\bigcup_{\mu\in\gamma} beta_\mu = \lambda$. In the last two cases, the inequality follows from the arguments above. If $\gamma = \lambda$, then $\omega(y)$ will have λ many pairs, and thus will be at least size λ.

Thus we find that $\omega(y_i) \in S(\alpha)$. □

Further, we prove the following fact:

Theorem 8. *For any surreal number with normal form* $\sum_{i \in \xi} \omega^{y_i} r_i$*, the sign sequence is given by*

$$\frown_{i \in \xi} (\omega^{y_i^o} r_i).$$

Let λ_i *denote the length of each summand, and* λ_i^o *denote the length of the each reduced sign sequence. Then the length of* $a = \sum_{i \in \xi} \lambda_i^o \leqslant \sum_{i \in \xi} \lambda_i$

Proof. This can be checked by induction. The only thing which needs to be verified is that $\lambda_i^o \leqslant \lambda_i$ for any given i. This is immediate, since omitting a '-' sign in the sign sequence means that the second argument of some pair drops from β_μ to some $\beta'_\mu \in \beta_\mu$, from which we will find

$$\omega^{\gamma_\mu + 1} \beta'_\mu \in \omega^{\gamma_\mu + 1} \beta_\mu,$$

which entails that

$$\sum_{i \leqslant \mu} \alpha_i + \beta'_i \leqslant \sum_{i \leqslant \mu} \alpha_i + \beta_i.$$

Thus, the sum of the lengths of the sequence

$$\sum \lambda_i^o \leqslant \sum \lambda_i.$$

□

We note that even if signs are omitted, equality may hold when summing sequences of a limit ordinal length.

Finally, we are able to prove the following:

Theorem 9. $(S(\alpha), +_{\text{No}}, \cdot_{\text{No}}) \models \text{CRing}$ *if and only if* $\alpha = \omega(\omega(\beta))$*, for some* $\beta \in \text{On}$*.*

Proof. In the forward direction, suppose that $S(\alpha)$ is a non-trivial characteristic zero commutative ring. Towards a contradiction, suppose that $\alpha = \sum_{i=I} \omega^{\alpha_i} n_i$, and let $x = \omega^{\alpha_0}$. It follows $x \in S(\alpha)$, and by hypothesis, $x^n \in S(\alpha)$ for all $n \in \omega$. But then $\omega\alpha_0 2 \in S(\alpha)$, which is impossible since $\omega^{\alpha_0 2} \ni \sum_{i \in I} \omega^{\alpha_i} n_i$.

In the converse direction, with $+, \cdot$ given as the standard surreal number constructions, if $\alpha = \omega(\omega(\beta))$ for some β, let $a, b \in S(\alpha)$. It suffices to check that $lh(ab) \in \omega(\omega(\beta))$.

Let $a = \sum_{i \in \lambda_1} \omega^{a_i} r_i$ and $b = \sum_{j \in \lambda_2} \omega^{b_j} s_j$. By Theorem ??? in [4],

$$ab = \sum_{(i,j) \in \lambda_1 \times \lambda_2} \omega^{(a_i + b_j)} (r_i s_j)$$

From Lemma 1 we have that $a_i, b_j \in S(\alpha)$ and from Theorem 7 that $a_i + b_j \in S(\alpha)$. Furthermore, we have that all real numbers are of length $\leqslant \omega$, and so all $r_i s_j \in S(\omega^{\omega\beta})$ for all $\beta \in \mathrm{On}$. Let $\lambda = o.t.(\lambda_1 \times \lambda_2)$. We check that $\lambda \in \omega^{\omega^\beta}$.

Finally, we note that at most $lh(a_i + b_j) = \max\{lh(a_i) + lh(b_j), lh(b_j) + lh(a_i)\} \in \omega^{\omega^\beta}$. This follows from $lh(a_i), lh(b_j) \in \omega^{\omega^\beta}$ with Cantor forms

$$\sum_{i=0}^{k_1} \omega^{\delta_i} n_i$$

$$\sum_{j=0}^{k_2} \omega^{\gamma_j} m_j$$

that $\delta_i, \gamma_j \in \omega^\beta$ for $i \in [k_1]$ and $j \in [k_2]$. Without loss of generality, further suppose that $\delta_i > \gamma_j$ for all i, j. Then we can relabel, δ_i, γ_j as ρ_l with $l \in [k_1 + k_2]$ with $\rho_i = \delta_i$ and $\rho_{l=k_1+j} = \delta_{l=k_1+j}$, and n_i, m_j appropriately relabeled. Then

$$lh(a_i + b_j) = \sum_{l=0}^{k_1+k_2} \omega^{\omega^{\rho_l}} n_l < \omega^{\omega^{\rho_0}}(n_0 + 1) \in \omega^{\omega^\beta}.$$

Thus

We now appeal to Theorem 8, and Lemma 1.

Thus $S(\alpha)$ will be closed under multiplication. □

What is remarkable here is that $\omega(\omega(\beta))$ are all prime ordinals. However, $\omega^\alpha + 1$ for all $\alpha \in \mathrm{On}$ are also prime, but do not correspond to algebraic objects. So not all primes classify algebraic theories of interest.

5.3 TODO $S(\alpha) \models$ Divisible Abelian Group

5.4 TODO $S(\alpha) \models$ ($\cdot$ Divisible Group)

5.5 TODO $S(\alpha) \models$ RCF

5.6 TODO $S(\alpha) \models$ RCVF

% * TODO Proof of Concept The proofs of concepts below are primarily for binary trees T of a given ordinal height α, such that T are full up α, and for proper subclasses of ordinals themselves.

In particular, $ht(T) = \alpha$, with T full up to α, then $T \cong \coprod_{\beta \in \alpha+1} \mathtt{P}(\beta) \cong \mathrm{No}(\alpha)$.

So the first full subtree that is an (ordered) additive group, i.e. closed under addition, will be of height ω, but each branch is of finite length. Precisely, this group is the dyadic rationals, which are also a ring. However, $\mathbb{Q}$, and $\mathbb{R}$ both are subtrees of height $\omega + 1$, a binary tree of height $\omega + 1$ is neither a group nor a field, since $\omega + \omega$ will be a branchof length $\omega 2$.

6 Maximal subtrees

6.0.1 Groups

Theorem 10. *A maximal initial subtree T of* No *is a group if and only if T is height $\omega(\alpha)$, for some $\alpha \in$* On.

Proof. In the forward direction, suppose that T is a maximal initial subtree of No is a group. Towards a contradiction, suppose further that T is of height $\gamma \notin \omega(\mathrm{On})$. In particular, γ has a Cantor normal form $\sum_{i \in I} \omega^{\alpha_i} n_i$, where indexing set I is such that $|I| > 1$ or $n_0 > 1$. Since $\alpha_i > \alpha_{i+1}$, $\omega^{\alpha_0} n_0$ will be an element in T. Moreover, since T is a group, $\omega^{\alpha_0} n_0 m$ will be an element in T for all $m \in \omega$. But since $\omega^{\alpha_0} n_0 > \sum_{i \in I \setminus \{0\}} \omega^{\alpha_i} n_i$, we have $\omega^{\alpha_0} n_0 2 > \sum_{i \in I} \omega^{\alpha_i} n_i$. Contradiction. Moreover, the same argument works when $|I| = 1$ and $n_0 > 1$.

In the converse direction, suppose that T is a maximal initial subtree of height $\lambda = \omega(\alpha)$ for some $\alpha \in$ On. Since $\omega(\alpha)$ is the simplest element in its respective Archimedean class, and every $x, y \in T$ is of length λ, by Fact 2.1, and $lh(x) + lh(y) < \lambda$, T is closed under addition. Since T is maximal, if $x \in T$, then $-x \in T$ by reversing all the signs in the sign sequence of x. The associativity and commutativity of addition has been studied exhaustively in [1, 3, 4]. Thus T is an (abelian) group. □

We can use the above theorem to also classify which maximal initial subtrees are ordered abelian groups given the work of [3].

We notice that $\mathbb{D}$ is not a divisible ordered abelian group, and so the first maximal initial subtree that is a non-trivial divisible ordered abelian group must be at some ordinal $\omega(\alpha)$ for $\alpha > 1$.

In fact, we can show that it must be $\epsilon(0)$ by considering the lengths of $\dfrac{\omega}{n}$. For all n, and $m < 2^n$ such that $(m, 2^n) = 1$, we have that $\dfrac{\omega m}{2^n}$ is a branch of length $\omega \otimes (n+1)$. So in particular, $\omega\mathbb{D}$ consists of branches of length $\omega^{<2}$. We now check find that $\omega\mathbb{Q}$ consists of branches up to length $\omega^2 = \omega(2)$, after which we repeat this procedure to find branchess of length $\omega(3)$, and so on, finding branches of $\omega(n)$ for all $n \in \omega$, whence the maximal initial binary tree of height ω^ω will be the first non-trivial divisible abelian group.

Claim. *$\omega\mathbb{Q}$ consists of branches of up to length ω^2.*

Proof. Suppose without lost of generality that $\frac{p}{q}$ has sign expansion $\langle \alpha_i, \beta_i \mid i \in \omega \rangle$, with $\alpha_i, \beta_i \in \mathbb{Z}^+$. Then since $\omega = \langle \omega, \varnothing \rangle$, the sign sequence of $\omega \frac{p}{q}$ will be

$$\langle \omega \oplus \omega \alpha_0^\flat, \omega\beta_0 \rangle \frown (\langle \omega\alpha_i, \omega\beta_i \rangle)_{0 \in i \in \omega}$$

Since $\frac{p}{q}$ is rational, the sign sequence of $\frac{p}{q}$ eventually repeats itself, and so there will be a maximal $N \geqslant \alpha_i, \beta_i$ for all $i \in \omega$. Thus $lh(\omega \frac{p}{q}) \leqslant \sum_{\omega} \omega * M = \omega^2$. □

Claim. *The maximal initial tree of height ω^ω is the first non-trivial divisible ordered abelian group.*

Proof. It is immediate that ω^ω will be closed under addition. We check that for any $a \in \{x \in \text{No} \mid lh(x) \in \omega^\omega\}$, there is $b \in \{x \in \text{No} \mid lh(x) \in \omega^\omega\}$ such that $a = b \cdot n$, i.e. $a\frac{1}{n} \in \{x \in \text{No} \mid lh(x) \in \omega^\omega\}$.

Towards that end, we use Theorem 5 from Section 1 to relate the normal form of $a = \sum_{i<\lambda} \omega^{a_i} r_i$ to its sign sequence $\frown_{i<\lambda} (\omega^{a_i^o} r_i)$. In effect, what we must show is that $\frown_{i<\lambda} (\omega^{a_i^o} \frac{r_i}{n})$ is of length $< \omega^\omega$ for all positive n.

In particular, each if a is of less than length ω^ω, then it is of length less than ω^N for some $N \in \mathbb{N}$, and moreover, each string $(\omega^{a_i^o} r_i)$ is of length less than ω^N, so $(\omega^{a_i^o} \frac{r_i}{n})$ will be of length less than ω^{N+1}, and thus $\frac{a}{n}$ will be of length less than ω^{N+2}, whence $\frac{a}{n}$ will be in $\{x \in \text{No} \mid lh(x) \in \omega^\omega\}$. $\square$

This leads to the next theorem

Theorem 11. *Let $\delta(x) := \{0, \omega(\delta(x^L) \cdot n)\} \mid \{\dfrac{\delta(x^R)}{2^n}\}$. Then a maximal binary tree T is a divisible ordered abelian group if and only if T is of height α for some $\alpha \in \delta\text{''}\,\text{On}$*

Proof. Not sure if this is correct. Working on it. $\square$

6.0.2 TODO More Structures

6.1 TODO Ordinals as algebraic objects

6.1.1 TODO Characteristic p

6.1.2 TODO p-adics

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