Foundations for the Analysis of Surreal-Valued Genetic Functions

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THESIS

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For Sam and David...

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SUMMARY

Conway's surreal numbers, No, is a class-sized field that was originally introduced for the analysis of certain two-person games, but which has proven interesting from multiple (including model-theoretic) points of view. For instance, Gonshor [1] showed that No can be equipped with a natural exponential function, which Ehrlich and van den Dries [2] showed makes No into a class-sized model of the real field with exponentiation. Considering the truncated surreal numbers of height less than some ordinal results in a set-sized structure; Ehrlich and van Den Dries [2] showed that the ordinals α such that truncation at α results in an exponential field that is a model of the reals with exponentiation were precisely the epsilon numbers. In recent years, various additional structures and functions on the surreal numbers have been considered in the literature (e.g. restricted analytic functions, derivations, logs, the ω map, valuations, etc) demonstrating that the surreal numbers form a universal object in the sense that set-sized models naturally embed as substructures.

In this thesis we systematize earlier results from the literature of functions on surreal numbers and consider the generalization of results of Ehrlich and van den Dries regarding models of $\mathbb{R}_{\text{exp}}$ to the wider class of *genetic functions*, which includes many examples of interest such as the class of restricted analytic functions, exp and log, as well as the ω map and other recursively definable functions. We do so by first amending the construction of arbitrary genetic functions given in [3], so that we may properly compose functions, and so that one can easily recover the definition of exp. We then analyze our newly proposed inductive construction with two natural notions of complexity, that of *generation*, which tracks the dependence on earlier genetic functions, and that of *Veblen rank*, which describes the complexity of subtrees closed under a genetic function, to characterize the ordinals α such that the surreal numbers below height α will correspond to models satisfying the cofinality conditions and the axioms of real closed fields.

After recovering fundamental analytic results for general surreal-valued functions, we further prove that every genetic function has a Veblen rank corresponding to an ordinal, and that our notion of Veblen rank behaves well under addition, multiplication, and composition, and in turn can be extended to arbitrary sets closed under said operations. In particular, the Veblen rank of a genetic function g identifies the largest subclass of epsilon numbers α such that sets of surreal numbers of height below α form a real closed field closed under g. From this, we establish many important functions, such as exp and log will have minimal Veblen rank. As a further consequence of our Veblen rank bound, we establish that every entire genetic function is *strictly tame* in the sense of Fornasiero [4]. Afterwards, with $\mathcal{G}$ denoting a set of genetic functions, we proceed to define a general first order theory $\mathsf{T}_{\mathcal{G}}$ whose models are $\mathcal{G}$-closed fields satisfying fundamental first order properties used to define each genetic function.

CHAPTER 1

INTRODUCTION

Conway [5] introduced the ordered field $\mathbf{No}$ of surreal numbers, which extends the field $\mathbb{R}$ of real numbers to a Class sized object as the minimal set-theoretic realization of a recursively defined game. Gonshor et al. [1,6,7] interpret this Class sized object as a lexicographically ordered binary tree that is endowed with a partial well-ordering $\leq_s$ such that a surreal number $\mathfrak{x}$ is a simpler than a surreal number $\mathfrak{y}$ if and only if $\mathfrak{x}$ may be regarded as an initial segment of $\mathfrak{y}$ when numbers are interpreted as members of the underlying tree. [6] show that $\mathbf{No}$ endowed with Kruskal's expoentiation function has the same elementary properties as the exponential field of real numbers, in addition to showing that when equipped with *restricted* analytic functions is also an elementary extension of the surreal numbers. Similar to the recursive construction of the surreal numbers, these aforementioned functions over the surreal numbers are recursively definable games, whose options correspond to definitions on arguments of lower complexity. Following the literature, such functions are called *genetic* functions. Despite appearing in the literature in the mid '70s with the publication of [5], no explicit formal definition for genetic functions appeared in the literature, until the early 2000's, first in Fornasiero's dissertation [8], which does not provide a base case for inductively constructing arbitrary genetic functions, and then Rubinstein-Salzedo and Swaminathan's paper *Analysis of Surreal Numbers* [3]. However, there are several deficiencies in the latter paper which are discussed and rectified in this dissertation. The most important rectification is introducing the additional condition of a cofinality property with respect to the pointwise *Dedekind representation* of the image of each function, which is expressible as a family of universal sentences. Once firmly establishing what constitutes a genetic function, and building off the work of Ehrlich et al [6, 9, 10], which established that the truncation of the surreal numbers to the full binary tree of height some epsilon number ϵ_γ, denoted by $\mathrm{No}(\epsilon_\gamma)$ for all ordinals γ will be a model for $\mathrm{RCF}, \mathbb{R}_{\exp}, \mathbb{R}_{\exp,an}$, we naturally ask if such a result extends to general RCF structures enriched with genetic functions.

We answer this question in the affirmative by introducing the notion of *Veblen rank*, which forestalls the possibility of any genetic function growing too quickly (such as one that sends the ordinals to singular cardinals). Furthermore, this notion of Veblen rank provides a tight characterization of the enriched RCF structures that are closed under finite application of the genetic functions appearing in a proper set of genetic function symbols $\mathcal{G}$. Briefly, we build off of Ehrlich and van den Dries' *weakened product lemma* [6] to define a pseudo-absolute value $\surd$ that tracks the leading term of the Cantor normal form of the length of a surreal number. We then use Gonshor's Fixed Point Theorem [1] and our revised definition of genetic functions to establish that the Veblen hierarchy of ordinal functions extends to a family of surreal-valued functions, and that this family is the natural family of functions for tracking the growth of complexity of arbitrary genetic functions. By combining these two ideas, we define our notion of Veblen rank as the union of *partial Veblen ranks*, corresponding to the image of maximum complexity of an element appearing in $g''\mathrm{No}(\epsilon_\gamma)$. We further extend our notion of Veblen rank to genetically definable inverse functions, including those that are not entire (see Lemma 14). We establish that each Veblen function φ_α has corresponding Veblen rank α, and also establish that the Veblen ranks of exp and log are both 0, as well as the functions h and g appearing in [1], which track the complexity of log and exp respectively. Further, we establish that the κ and λ maps used in [7,11] to study the *exp-log numbers* and the *log-atomic numbers* respectively (and ultimately to define the Berarducci-Mantova derivative ∂_{BM}) both have Veblen rank 1 (see Props 29 and 31).

Additionally, we show that every genetic function is *strictly tame* in the sense of Fornasiero [4] (see Theorem 54). We also proceed to define $Q_{\mathcal{G}}$ as the definable field over $\emptyset$ in the language $\mathcal{L}_{or} \cup \mathcal{G}$, where $\mathcal{G}$ is a set of genetic functions closed under ancestors. We then define a corresponding theory $T_{\mathcal{G}}$ as a extension of RCF which preserves the inductive properties of each $g \in \mathcal{G}$ (e.g. if $g \in \mathcal{G}$ is monotonic, or injective, and if $g(h(x, y)) = k(g(x), g(y))$ for $g, h, k \in \mathcal{G}$), the family of universal sentences corresponding to the cofinality property of each $g \in \mathcal{G}$, and the preservation of the definable ordering of the field $Q_{\mathcal{G}}$. It follows that models of $T_{\mathcal{G}}$ can be interpreted as ordered $Q_{\mathcal{G}}$ vector spaces. We proceed to establish that given $VR(\mathcal{G}) = \alpha$, $\mathrm{No}(\varphi_{\alpha+1}(\gamma)) \models T_{\mathcal{G}}$, and consequently $\mathrm{No} \models T_{\mathcal{G}}$. This will set-up (while leaving for future papers) how to interpret models of $T_{\mathcal{G}}$ as $\mathcal{G}$-structured Hahn fields, and generalize the result of Ehlrich-Kaplan [12] that models of $T_{\mathcal{G}}$ are isomorphic to an initial substructure of No if and only if the model is isomorphic to a truncation-closed, cross-sectional $\mathcal{G}$-structured subfield of a $\mathcal{G}$-structured Hahn field.

1.0.1 Background

Deep connections between various abstract two-player games and logic abound in mathematical literature. Games have been used to study set theory; descriptive set theorists use games to study Baire space and complexity; Henkin used games to give a semantics for infinitary languages; Ehrenfeucht-Fraïssé games are used to build partial embeddings between structures in classical model theory, and in general, Hodges and others study how games can be used to produce families of pairwise non-isomorphic structures.

One particular class of abstract games rife with examples of interest for logicians is the class of *combinatorial games*. These are abstract two-player games with no hidden information, and no chance elements, and which are recursively defined in terms of simpler games. The two players of these games are traditionally named Left and Right, who play alternately, and whose moves affect the *position* of a game in a manner determined by the rule set of a game.[1]

The Class of **partizan games**[2], denoted by PG, which distinguishes the two players by the moves available to them[3], ought to be of particular interest to logicians. Conway [5] established that this class of games can be recursively endowed with a partial ordering relation and with a notion of addition, which endows PG with the structure of a partially ordered abelian group. Following the work of [13–15], PG is the universal homogeneous embedding model for the

[1] Throughout this document, the term game refers to an individual position in a combinatorial game, and the available moves to a given player are referred to as Left and Right options respectively. Each option is a direct move from the current game position to a new game position, which in turn affects the available options for the next player.

[2] Although the formal study of *partizan games* can be said to have been underway when Zermelo first analyzed chess at the turn of the 20^{th} century, the Class of partizan games was not defined in earnest until Conway, Berlekamp, and Guy began to study them in the latter half of the 20^{th} century. This Class contains many familiar games, like Go and Chess.

[3] Specifically, Left and Right have distinct, non-identical option sets.

theory of partially ordered abelian groups. These include ℓ-groups, and interval effect algebras, which interpret the semantics of deductive systems.

The recursive construction of the class of partizan games, of $\leq$, and of $+$ also leads to a *distinguished* subclass: the surreal numbers. By taking the restriction of this partial order to a maximal linear order closed under $+$, one recovers an ordered abelian group, the surreal numbers, denoted here by No. One can further recursively define a notion of multiplication restricted to the elements of this Class sized order group, and endow it with the structure of a saturated real-closed field. The recursive process outlined in [1,5,16] for defining multiplication, and in [1] for defining exp, imitates the game construction of [1,5], and a function constructed this way has been called a genetic function.

Throughout this dissertation, our primarily interest is in the global surreal valued functions that can be defined with respect to a recursive construction whose unique value is expressible in terms of a partizan game respecting order and uniformity conditions. Precisely, the Left and Right options of the genetic function are defined with respect to surreal numbers and genetic functions of lower complexity than the term presently being evaluated. In this dissertation, we make these notions precise in general by building off the work of Gonshor, Ehrlich, van den Dries, and others.

Although the literature is rife with genetic functions, there have been scant few that actually attempt to define genetic functions in general. Specifically the seminal work of [3,4,8] provides the basis of a general definition for a surreal-valued genetic function. My modest contribution here is to correct for minor defects in [3], described below, and make explicit the fundamental relationship between uniformity and simplicity that allows us to recursively construct surreal-valued functions as one recursively constructs surreal numbers.

In addition to providing a rigorous definition for genetic functions, this dissertation is structured around proving the following two main results:

(I) Every surreal-valued genetic function $\mathfrak{g}$ is of bounded complexity, which we identify with the newly defined notion of *Veblen rank*;

(II) For every set of genetic functions $\mathcal{G}$, we can define a theory $\mathsf{T}_{\mathcal{G}}$ such that all models of $\mathsf{T}_{\mathcal{G}}$ are real closed fields closed under the genetic functions of $\mathcal{G}$, the functions and their term sets are comprehended, and there is a corresponding definable ordered field $\mathsf{Q}_{\mathcal{G}}$ such that we may regard our model as an ordered $\mathsf{Q}_{\mathcal{G}}$ vector space. Specifically, using our notion of Veblen rank, cutting off the surreal numbers at the height corresponding to the Veblen rank will produce a model of $\mathsf{T}_{\mathcal{G}}$, from which we have that No is also a model.

This approach in turn ought to serve as the basis for defining genetic functions for other distinguished convex partially ordered abelian groups.

The approach considered in this dissertation is an attempt to reconcile the differing approaches to defining the surreal numbers in the literature, and in particular the subtle distinction between a partizan game being in canonical form, that is, of minimal set theoretic rank and options, and a surreal number in the sense of Gonshor having a canonical form being expressible in terms of a **sign sequence**, which are equivalently understood as a functions from some ordinal α corresponding to the minimal set theoretic rank of the number $\mathfrak{a}$ to an ordered two-value set whose members are typically denoted $\{\ominus, \oplus\}$, with the following ordering convention on the elements $\ominus < \emptyset < \oplus$ extending to a lexicographical ordering on all sign sequence representations of surreal numbers. In particular, one can reconstruct on the nose the canonical options of a surreal number given its sign representation and vice versa.

The subtle distinction here is that the latter can be said to be the **position closure** of the canonical form (see Chapter 2.1 for further details). This distinction is important, as we find that the uniformity condition we wish to impose must extend to the maximal position closed Class of options (see Chapter 2.4 for further details), which following [3] we call the *Dedekind representation*. Using recursively define Conway cuts for option terms satisfying certain properties that extend to the Dedekind representation of the function output, we can then identify the initial subtrees satisfying the properties of the function and further ensure that all such functions are closed under composition (when composed with other entire genetic functions).

Curiously, Conway expresses his dismay in the Epilogue of the second edition of [5] regarding the development of research into the surreal numbers in terms of sign sequences and the importance of researching genetic definitions in the general sense, as follows:

> This has the great advantage of making equality be just identity rather than an inductively defined relation, and also of giving a clear mental picture from the start. However, I think it has probably also impeded further progress. Let me explain why.
>
> The greatest delight, and at the same time, the greatest mystery, of the Surreal numbers is the amazing way that a few simple "genetic" definitions magically create

> a richly structured Universe out of nothing. Technically, this involves in particular the facts that each surreal number is repeatedly redefined, and that the functions the definitions produce are independent of form. Surely real progress will only come when we understand the deep reasons why these particular defintions have this property? It can hardly be expected to come from an approach in which this problem is avoided from the start?
>
> The sign-sequence definition has also the failing that it requires a prior construction of the ordinals, which are in ONAG produced as particular cases of the surreals. To my mind, this is another symptom of the same problem, because the definitions that work universally should automatically render such prior constructions unnecessary. There is also a peculiar emphasis on the number $\frac{1}{2}$ that is totally unnecessary – in ONAG $\{\frac{1}{3}|\frac{2}{3}\}$ is just as good a definition of $\frac{1}{2}$ as $\{0|1\}$ is – and that I think serves to obscure the underlying structure.
>
> I believe the real way to make "surreal progress" is to search for more of these "genetic" definitions and seek to understand their properties.

This complaint is curious because in correspondences with Ehrlich [17] Conway expressed regret in terms of how he expressed the inductive structure of numbers in terms of birthdays and ambiguity over what was meant by x being simpler than y. The sign sequence approach and the corresponding binary tree has the benefit of making the concept of simplicity explicit: a surreal number (sign sequence) x is simpler than a surreal number (sign sequence) y if it is a proper initial segment of y.

The author proposes that the surreal numbers form a *distinguished group* in the following sense: they form a convex subgroup which can be endowed with a notion of commutative multiplication, and the canonical form of elements of the group (those of minimal set theoretic rank) correspond to members in a tree such that the initial segments of each branch correspond to the options of the position closure of the canonical form of the game. What distinguishes this group from other substructures of PG is that one can define polynomials and by extension, genetic functions, using the game notation, which can equivalently be understood as the unique minimal set-theoretic realization of a partial type; in the case of the surreal numbers, the unique minimal realization of a cut (see Chapter 2.4.0.1 for further details).

To make all of this explicit, we now provide the sign sequence definition of surreal numbers:

Definition 1. *A* ***surreal number*** $\mathfrak{a}$ *is a function from an initial segment of the ordinals* On *into the two valued set* $\{\ominus, \oplus\}$ *[1]. The surreal numbers can be lexicographically ordered with the rule that* $\ominus < \emptyset < \oplus$*, and can be informally understood as an ordinal length sequence consisting of pluses and minuses which terminate. We call such sequences* ***sign sequences****. This includes the empty sequence.*

In turn, the ***surreal numbers****,* No*, can be identified with the binary tree*

$$\bigcup_{\alpha \in \mathrm{On}} {}^{\alpha}2 = 2^{<\mathrm{On}},$$

which can be endowed with the aforementioned lexicographical linear ordering.

Remark 1. *Gonshor's construction is contrary to the construction presented in [5]; the principle advantage of adopting Gonshor's definition is that we are able to reason directly about the complexity of a surreal number without also needing to consider the equivalence class to which the number belongs as in the Conway construction.*

Furthermore, we are able to introduce the following function:

Definition 2. *Let* $\iota : \mathrm{No} \to \mathrm{On}$ *denote the function which determines the domain of a surreal number, i.e. for* $a \in \mathrm{No}$, $\iota(a) = \alpha$, *such that* $a : \alpha \to \{\ominus, \oplus\}$.

Remark 2. *While Gonshor refers to* ι *as the* ***length*** *of a surreal number, given that there is a corresponding sign sequence, Ehrlich refers to* $\iota(a)$ *as the* ***tree-rank*** *of a surreal number* a. *This is directly related to the notion of the* birthday of a game, *introduced by Conway and Knuth, which will be examined in more detail in Chapter 2.1.*

As mentioned above, via an induction argument, there is a bijective correspondence between the sign sequence expansion of a surreal number, denoted by (a), and the understanding of $a : \alpha \to 2$, where $2 = \{\ominus, \oplus\}$. Supposing this is true below α, and given a surreal number as binary function, $a : \alpha \to 2$, we can consider the sequence $(a_\beta)_{\beta \in \alpha} = (a \restriction \mathtt{beta}))_{\beta \in \alpha}$. Each $a_\beta \sqsubset a$, and further if $a(\beta) = \ominus$, then a_β is equivalent a canonical Right option of a, and similarly if $a(\beta) = \oplus$, then a_β is a canonical Left option of a. In particular, we have this ordering considering the base case of a_0. Here $a_0 = \emptyset$, while $a(0)$ indicates whether a is positive or negative.

In general, we'll be collecting the signs $\oplus$ and $\ominus$ in pairs of blocks, denoted by α and β respectively, e.g., the sign sequence expansion $(a) = (\langle \alpha_i, \beta_i \rangle_{i \in \phi a}$ where ϕa denotes the order type of non-trivial blocks of $\langle \alpha_i, \beta_i \rangle$. Specifically, $\langle \alpha_i, \beta_i \rangle$ indicates that with $\gamma = \bigoplus_{j<i}(\alpha_j \oplus \beta_j)$, the sign sequence expansion of a can be broken down to the sign sequence expansion of

$$(a \restriction \gamma) \frown \oplus \frown \oplus \frown \cdots \frown \oplus \frown \ominus \frown \ominus \frown \cdots \frown \ominus =: (a \restriction (\gamma \oplus \alpha_i \oplus \beta_i),$$

with α_i many $\oplus$ symbols followed by β_i many $\ominus$ symbols. If, for example, α_i were at least the size of some limit ordinal λ, this would mean that we have a limit ordinal subsequence of $\oplus$ symbols in the sign sequence expansion, or correspondingly, starting at $\gamma \in \alpha$, there are at least λ many ordinals below α such that for $\gamma \in j \in \gamma + \lambda$, $a(j) = \oplus$.

Further, by judicious application of the rules of ordinal arithmetic, we may take infinite pairwise sums of the pairs $\langle \alpha_i, \beta_i \rangle$, to determine the *length* (or corresponding birthday, or tree-rank of a surreal number). This is because the given infinite sums can be inductively computed over the ordinal ϕa as partial sums. In particular, we must be mindful of absorption, whereby if β_i is greater than or equal to a limit ordinal containing α_i, then $\alpha_i \oplus \beta_i = \beta_i$, or if for a limit ordinal $\lambda \in \phi a$ we have each $\alpha_i, \beta_i < \lambda$ for all $i \in \lambda$, then the corresponding sum will be λ at λ.

With these rules in place, one can interpret the sequences of these blocks as members in a binary tree. In turn, by identifying sign-sequences with members of a binary tree, we can interpret the Class of all sign sequences as a lexicographically ordered binary tree in the spirit of Ehrlich [9, 17]. Specifically, we have the following definitions:

Definition 3. *Let* $<_s$ *denote the* ***simplicity*** *partial order, so-called because each surreal number can be assigned to an ordinal indicating the level of recursion via the* birthday function *(equivalently, the* ι *mapping) so that* x *is said to be simpler than* y *if x is born prior to y, e.g* $\iota(x) < \iota(y)$*, and* x *is an initial segment of* y*.*

Equivalently, x *is* ***simpler than*** *y if and only if* x *is a proper initial segment of* y*.*

Equivalently, when regarding surreal numbers as functions from ordinals to a two-valued set, we say that x *is simpler than* y *if there is some ordinal* $\alpha \in \iota(y)$ *such that* $y \restriction \alpha = x$*.*

Equivalently, using the game notation of the cut construction, supposing that x, y *are surreal numbers such that* $x = X|Y$*, where* X *is used to denote the set of* left options *of* x *(the moves available to the Left player), and* Y *is used to denote the set of* right options *of x (the moves available to the Right player), and* $X < x < Y$*, we say* x *is* simpler in the tree-theoretic sense than y *whenever* $X < y < Y$ *and* $x \neq y$*.*

A tree $\langle A, <_s \rangle$ *is a partially ordered Class such that for each* $x \in A$*, the Class of predecessors* $\mathrm{pr}_A(x) = \langle y \in A \mid y <_s x \rangle$ *is well-ordered by* $<_s$*, and* x *is said to have* ***tree-rank*** α*, denoted by* $\rho_A(x) = \alpha$*, where* α *is the order type of* $\mathrm{pr}_A(x)$*. Precisely, the birthday function (equivalently, the* ι *mapping) assigns each surreal number to the* ***tree-rank*** *of its set of predecessors.*

A ***root*** *of a tree* A *is a member of the zeroth level, the* α *level of a tree A is the Class* $\mathrm{Lev}_A(\alpha) = \{x \in A \mid \rho_A(x) = \alpha\}$*.*

If A *is a tree, and* $x, y \in A$*, then y is an* ***immediate successor*** *of x if* $x <_s y$ *and* $\rho_A(y) = \rho_A(x) \oplus 1$*. If* $(x_\alpha)_\beta$ *is a chain ordered by* $<_s$ *of order type* β*, then* y *is an immediate successor of the chain if* $x_\alpha <_s y$ *for all* $\alpha \in \beta$ *and* $\rho_A(y) = \inf\{\gamma \in \mathrm{On} \mid \forall \alpha \in \beta . \gamma > \rho_A(x_\alpha)\}$*.*

A binary tree $\langle A, <_s \rangle$ *is **full** if every element has two immediate successors and every empty chain or of limit ordinal length has an immediate successor.*

A binary tree $\langle A, <_s, < \rangle$ *is **lexicographically ordered** by the order relation* $<$ *if for every* $x, y \in A$*,* x *is* $<_s$ *incomparable with* y *if and only if there is a common* $<_s$ *ancestor* $z <_s x, y$ *such that* $x < z < y$ *or* $y < z < x$*. A lexicographically ordered binary tree* $\langle A, <_s, < \rangle$ *is **complete** if whenever* L *and* R *are subsets of* A *such that* $L < R$*, then there is a* $y \in A$ *such that* $L < \{y\} < R$*. All s-hierarchical structures appearing in [9, 10, 17–19] are said to be complete whenever they are complete as lexicographically ordered trees. A binary tree* A' *is said to be an **initial subtree** of a binary tree* A *if* $A' \subseteq A$ *and the induced ordering is such that for all* $x \in A'$*, the set of predecessors* $\mathrm{pr}_{A'}(x) = \{y \mid y <_s x\} = \mathrm{pr}_A(x)$*.*

A class of $A \subset \mathrm{No}$ *is **convex** if and only if for all* $x, y \in A$ *and for all* $z \in \mathrm{No}$*, if* $x < z < y$*, then* $x \in A$*.*

To make the notion of simplicity precise, the surreal numbers can be considered members of a binary tree endowed with a partial order relation $<_s$ corresponding to $\sqsubset$, and an order relation $<$ recursively defined by $\ominus < \emptyset < \oplus$. In this sense, the base language for any theory of surreal numbers consists of $\mathcal{L}_{\mathsf{base}} = \{<, <_s, 0\}$ [9, 10, 17–19], with T_{base} a theory of a lexicographically ordered binary tree with 0 as the $<_s$ minimal element. However, many important and interesting properties of the surreal numbers cannot be expressed within First-order logic.

As is customary, 0 is the unique surreal number whose domain is the empty set, and from this it is immediate that $\iota(x) = 0$ if and only if $x = 0$. Further, it is known that $\iota(x+y) \leq \iota(x) + \iota(y)$

by induction on $\iota(x), \iota(y)$ (see [1] for details). However, it remains unknown whether ι is pseudo-absolute value; Gonshor conjectures in [1] that $\iota(xy) \leq \iota(x)\iota(y)$.

As of the time of this writing, a proof of this product inequality remains unpublished; the status of this result having been unpublished was noted in [2], where the authors work around this by proposing a weaker product inequality:

$$\iota(xy) \leq \omega\iota(x)^2\iota(y)^2.$$

It can be shown that the surreal numbers constructed by Gonshor are the canonical elements of the corresponding equivalence class of a surreal number in the sense of Conway. However, throughout this thesis we will prefer to explicitly work with Gonshor's construction given that we may straightforwardly induct on the length/tree-rank of surreal arguments. We nonetheless will appeal to the Conway cut construction, to be defined below, by identifying the minimal length sequence satisfying a cut, as the *simplest* game, as in Ehrlich.

We can summarize the properties of $(\mathrm{No}, \leq, \leq_s)$ with the following theorem:

Theorem 1. *1.* $(\mathrm{No}, \leq) \models \mathrm{DLO}$

2. $(\mathrm{No}, \leq_s) \models \mathrm{WFPO}$ *(Well-founded partial order)*

3. For every $a \in \mathrm{No}$*,* $\mathcal{S}(a) := \{x \in \mathrm{No} \mid a \leq_s x\}$ *is a convex subclass of* No*;*

4. Whenever A *is a non-empty convex subclass of* No*, there is a unique simplest element* $a \in A$ *such that for all* $x \in A$*,* $a \leq_s x$*;*

5. *For strict subsets* $L, R \subset \mathrm{No}$ *such that* $L < R$, *the corresponding type* $p_{L|R}(x) := \{x \in \mathrm{No} \mid L < x < R\}$ *is a non-empty convex class with simplest element* $a = L|R$, *the* cut-realization *of the Conway construction.*

Additionally, Conway established that surreal numbers a can be expressed in a normal form

$$a = \sum r_i \omega^{y_i}$$

where $r_i \in \mathbb{R}$ and (y_i) forms a well-ordered descending sequence.

Throughout this dissertation, the following Greek letters are used denote functions describing surreal numbers:

Definition 4.

- $\alpha : \mathrm{No} \to (\mathrm{On} \to \mathrm{On})$ *such that* $\alpha(a)(i) = \alpha_i(a) =$ *the ordinal number of* $\oplus$ *symbols in the* i^{th} *pair of sign symbols in the sign sequence of a;*
- $\beta : \mathrm{No} \to (\mathrm{On} \to \mathrm{On})$ *such that* $\beta(a)(i) = \beta_i(a) =$ *the ordinal number of* $\ominus$ *symbols in the* i^{th} *pair of sign symbols in the sign sequence of a;*
- $\gamma : \mathrm{No} \to (\mathrm{On} \to \mathrm{On})$ *where* $\gamma(a)(i) = \gamma_i(a) = \bigoplus_{j \le i} \alpha_j(a)$.
- $\phi : \mathrm{No} \to \mathrm{On}$ *describes the number of non-trivial pairs in the sign sequence of a surreal number.*
- $\nu : \mathrm{No} \to \mathrm{On}$ *describes the number of non-trivial summands in the Conway normal form of a surreal number, to be described in the following definition. In particular,* ν *describes the order type of the* ***support*** *of a surreal number.*

- $\iota : \mathrm{No} \to \mathrm{On}$ *describes the* ***length*** *(alternately,* ***tree-rank****) of a surreal number which can be computed as follows:*

$$\iota(a) = \bigoplus_{i \leq \phi a} (\alpha_i(a) \oplus \beta_i(a))$$

where in the equation above, addition is understood to be ordinal addition and not surreal addition.

To reiterate, No can be recursively built in the following three equivalent ways:

Definition 5. *1. Sign sequence: Given a surreal number a with tree-rank α, we can decompose $a : \alpha \to 2$ as a sequence of pairs:*

$\langle \alpha_i, \beta_i \rangle$ where α_i, β_i are ordinals, such that $\alpha_i = 0$ only if $i = 0$ or i is a limit ordinal, or for some $j \leq i$, for all $k \geq j$, $\alpha_k = 0$, and $\beta_i = 0$ if and only if there is some $j \leq i$ such that for all $k \geq j$, $\beta_k = 0$. We then concatenate these pairs to describe the ***sign sequence*** *of a as*

$$(a) = \frown_{\phi a} \langle \alpha_i(a), \beta_i(a) \rangle$$

2. Cuts[1]:

Cuts[1]:

Cuesta-Dutari *Historically this was the first development of what can be identified as the surreal numbers.*

---

[1] *As we shall see in Chapter 2.1, the partial ordering relation of partizan games emerges from a condition on the sets of left and right options which define a game. For this reason, definition with respect to cuts only applies to a recursive construction of surreal numbers and not the broader Class of games itself. In the broadest possible sense, surreal numbers are recursively defined with respect to loop-free partizan games that are totally ordered by the partial order relation defined for games.*

We let X *denote an ordered (not necessarily proper) Class in* NBG*, and denote by* (L,R) *a disjoint pair such that* $\mathsf{L}\cup\mathsf{R}=\mathsf{X}$ *and* $\mathsf{L}<\mathsf{R}$*.*

Definition 6. $\mathcal{C}(\mathsf{X})=\{(\mathsf{L},\mathsf{R})\colon \mathsf{L}\cup\mathsf{R}=\mathsf{X}\wedge\mathsf{L}<\mathsf{R}\}$

It is immediate that $\mathcal{C}(\mathsf{X})$ *is non-empty for every ordered Class* X *(including the empty set).*

Definition 7. *Let* $\chi(\mathsf{X})=\mathsf{X}\cup\mathcal{C}(\mathsf{X})$ *denote the* ***Cuesta Dutari completion*** *of* X*, ordered by*

(a) if $x,y\in\mathsf{X}$*, then* x *and* y *are ordered as in* X*;*

(b) if $x\in\mathsf{X}$ *and* $y=(\mathsf{L},\mathsf{R})\in\mathcal{C}(\mathsf{X})$*, then* $x<y$ *if* $x\in\mathsf{L}$ *and* $y<x$ *if* $x\in\mathsf{R}$*;*

(c) if $x=(\mathsf{L},\mathsf{R})$ *and* $y=(\mathsf{F},\mathsf{G})$ *in* $\mathcal{C}(\mathsf{X})$ *such that* $\mathsf{L}\neq\mathsf{F}$*, then* $x<y$ *if* $\mathsf{L}\subsetneq\mathsf{F}$*, o.w.* $y<x$*.*

It is a routine proof by cases to verify that $\chi(\mathsf{X})$ *is an ordered Class. It is further an easy exercise to verify that* $\chi(\mathsf{X})$ *contains its infimum and supremum, at Cuesta-Dutari cuts* $(\mathsf{X},\emptyset)$ *and* $(\emptyset,\mathsf{X})$*. Finally, we have the following proposition:*

Proposition 1. *For ordered Class* X

(a) For all $x<y$ *in* X*, there is* $c\in\mathcal{C}(\mathsf{X})$ *such that* $x<c<y$*;*

(b) For all $c<d\in\mathcal{C}(\mathsf{X})$ *there is* $x\in\mathsf{X}$ *such that* $c<x<d$*;*

Proof. Following the proof found in Chapter 4.02 of [16], for (1), let $\mathsf{L}=\{t\colon t\leq x\}$ and $\mathsf{R}=\{t\colon x<t\}$. Then by the ordering established above, $x<(\mathsf{L},\mathsf{R})<y$. For

item (2), if $c < d$, then with $c = (L, R)$ and $d = (F, G)$, it follows by the ordering established above that $L \subsetneq F$, so we may choose $x \in F \backslash L$. $\square$

We can recover the surreal numbers as follows:

Let $X_0 = \emptyset$. *Then define* $X_{\alpha+1} = \chi(X_\alpha)$ *and for limit ordinals* λ, *let* $X_\lambda = \bigcup_\lambda X_\alpha$.

By induction $(X_\alpha)_{\alpha \in \mathrm{On}}$ *is defined as a transfinite increasing chain ordered by inclusion.*

Following [5], denote by $O_\alpha = X_\alpha$, *which we'll identify with the numbers defined before day* α. *Set* $N_\alpha = X_{\alpha+1} \backslash X_\alpha$ *to denote the new numbers made on day* α, *and finally set* $M_\alpha = X_{\alpha+1} = O_\alpha \cup N_\alpha$, *denoting all the numbers made by* α. *Following these identifications, we find*

$$\mathrm{No} = \bigcup_{\alpha \in \mathrm{On}} O_\alpha = \bigcup_{\alpha \in \mathrm{On}} M_\alpha.$$

We note that $(N_\alpha)_\alpha$ *partitions* No. *In fact,* $N_\alpha = \mathsf{Lev}_{\mathrm{No}}(\alpha)$, *i.e. the numbers with tree-rank* α. *Further, our birthday/tree-rank function identifies for each* $x \in \mathrm{No}$ *the least* $\beta \in \mathrm{On}$ *such that* $x \in M_\beta$.

Conway Cuts *Given two* sets F, G *of surreal numbers such that every element of* F *is less than every element of* G, *we can define a surreal number as the* simplest[1] *number* c *such that* $F < c < G$, *i.e. for any* $d \in \mathrm{No}$ *such that* $F < d < G$, $c \sqsubseteq d$. *For the corresponding*

[1] *Simplicity in the sense of [17] is identical to the minimal length requirement of [1].*

number (game) c*, we call* F *and* G*, the* ***option sets of c****, with* F *denoting the* ***Left options*** *and* G *denoting the* ***Right options****.*

In this sense, c *is a cut with representative sets* F, G*. Every surreal number has a unique* ***canonical cut*** *consisting of sets* L_c *and* R_c *such that every element of* $L_c, R_c <_s c$*, and for every cut* $c = F|G = L_c|R_c$*,* F *must be cofinal in* L_c *with respect to* $<$ *and* G *must be coinitial in* R_c *with respect to* $<$ *(Theorems 2.8, 2.9 [1]).*

3. *Conway normal form: surreal numbers can also be described as formal sums with base* ω *in their Conway normal form. Precisely,* $a = \sum_{\nu a} \omega^{a_i} r_i$*, where* $(a_i)_{\nu a}$ *forms a descending sequence of surreal numbers* a_i*, while for each* $i \in \nu a$*,* $r_i \in \mathbb{R}^\times$*. Additionally, we can describe surreal numbers in Ressayre normal form, with base respect to* exp*.*

Remark 3. *Because our results will need to move freely between* ordinal *addition and* surreal *addition, we will prefer to write ordinal sums as* $\bigoplus_i \omega^{\alpha_i}$ *to distinguish from surreal sums* $\sum_i \omega^{\alpha_i}$*. We further note that* $\bigoplus_{i \leq n} \omega^{\alpha_i} = \sum_{i \leq n} \omega^{\alpha_i}$ *whenever* (α_i) *form a descending sequence.*

Throughout this dissertation, every operation of interest is defined **genetically** vis a vis the $<_s -$ minimal realization of recursively constructed partial types. This leads to the first, naive, definition of genetic function, although we will revisit and expand upon this definition in Chapter 3, where we make clear the precise ground functions on which we are recursing.

Definition 8. *We say that the sets of formula appearing in the left and right option sets of the Conway cuts are **genetic formula**.*[1] *We naively*[2] *say* f *is a **genetic function** if it is a recursively defined function with respect to cuts that possesses the **uniformity property**.*

Any recursively defined operation with respect to cuts has the uniformity property in the sense of [1] if the formulas do not depend on a given representation of the specific arguments. Explicitly, following [4] this means given arbitrary left options $f^L(x, y, z; f(y), f(z))$ *and right options* $f^R(x, y, z; f(y), f(z))$ *where the function symbols appearing in* f^o *are not necessarily evaluated on* y *or* z*, we have the following two conditions holding:*

<u>*Gonshor Uniformity*</u> *for* $x \in \mathrm{No}$ *such that* $x = G|H$ *where* $G|H$ *is any representation of* x *with left elements* g *and right elements* h*, then*

$$f(x) = \left\{f^L(x, g, h)\right\} | \left\{f^R(x, g, h)\right\} = \left\{f^L(x, x^L, x^R)\right\} | \left\{f^R(x, x^L, x^R)\right\}$$

with f^L, f^R *ranging over terms from the option sets.*

<u>*Global cofinality*</u> *for all* $x, y, z \in \mathrm{No}$ *such that* $y < x < z$

$$f^L(x, y, z; f(y), f(z)) < f(x) < f^R(x, y, z; f(y), f(z))$$

---

[1] *Technically, these should be called **genetic terms**; the genetic relation here is the one saying that the realization of the formula satisfies* $t^L < x < t^R$*, where* t^L *and* t^R *are generic terms from the left and right sets respectively, and* x *is the simplest surreal number satisfying this order relation.*

[2] *This definition is naive because we do not yet clarify the nature of this recursive construction. This definition is being provided here to emphasize the properties that we need to have satisfied, and not just the template for constructing genetic functions.*

for all Left option terms f^L *and all Right option terms* f^R*;*

For example, addition is commonly given with respect to the canonical representatives of the addends, i.e.

$$a + b = \left\{a^L + b, a + b^L\right\} | \left\{a^R + b, a + b^R\right\},$$

but uniformity holds that if $a = F|G$ and $b = H|K$, then

$$a + b = \{f + b, a + h\} | \{g + b, a + k\} = \left\{a^L + b, a + b^L\right\} | \left\{a^R + b, a + b^R\right\}.$$

When analyzing functions with the uniformity property, unless otherwise stated, we will let a^L, a^R denote arbitrary elements of the left and right set of the canonical representation of a. Furthermore, these left and right sets precisely correspond to the Left and Right options of a game.

We will be returning to and expand this definition in Chapter 55. We are primarily concerned throughout this dissertation with the following:

- providing a rigorous definition for genetic functions, specifically surreal-valued genetic functions;
- providing a measure of complexity for surreal-valued genetic functions;
- providing a means of constructing a theory for sets of genetic functions such that No is a model of said theory, and relevant model theoretic characterizations of No.

The following theorem lists a few examples of genetic functions in the literature [1–3, 16]

Theorem 2. *Addition, the successor function, multiplication,* ω*,* ϵ*,* exp*,* log*, and* arctan*, are all genetic functions on some convex class of* No*. In particular, addition, the successor function, multiplication,* ω*,* ϵ *and* exp *are entire genetic functions.*

One important notion to make the definition of a surreal-valued genetic function rigorous is that it's not simply recursively with respect to option sets, but that these option sets possess the *uniformity property* discussed in [1]. The following theorem provides a specific example for why the uniformity property is desirable:

Theorem 3. *We may recursively define* ι *with respect to the canonical cut of a surreal number, but* ι *does not have the uniformity property, and is therefore not a genetic function.*

Proof. This is immediate. We can define $\iota(a) = \left\{\iota(a^L), \iota(a^R)\right\} | \{\}$. This definition is recursively defined with respect to canonical cuts, always returns an ordinal value, but it fails to be uniform. Consider $a \in \text{No}^\times$, so $\iota(a) > 0$. Then

$$\begin{aligned} 0 &= \iota(a + (-a)) \\ &= \left\{\iota(a^L + (-a)), \iota(a + (-a)^L), \iota(a^R + (-a)), \iota(a + (-a)^R)\right\} | \{\} \\ &= \left\{\iota(a^L - a), \iota(a - a^R), \iota(a^R - a), \iota(a - a^L)\right\} | \{\} \\ &= \iota(a) + 1 \\ &> 0. \end{aligned}$$

□

1.1 Summaries

The main results of this dissertation follow from the work of Ehrlich and van den Dries [2], whereby they establish that the natural (genetic definition) of exponentiation given in [1] (and credited to Kruskal), endows No with the structure of a real-closed field with exponentiation. In [2], the authors established several bounds on the complexity of surreal numbers and exp as a genetic function. In particular, they identified initial subtrees of the surreal numbers consisting of numbers whose height is below epsilon numbers α are real-closed exponential fields, and that further, it is a necessary and sufficient condition that α be an epsilon number for initial subtrees consisting of elements of heights below α to be models of real-closed fields,[1] as well as real-closed subfields with exponentiation. The author of this dissertation noticed that since epsilon numbers are fixed points of a genetic function ω, and that higher epsilon numbers correspond to both the Veblen functions in the Veblen hierarchy, as well as genetic functions guaranteed by Gonshor's fixed point theorem [1], that said higher epsilon numbers ought to provide a good measure for the complexity of genetic functions in general, as well as indicating some natural models for a given theory of genetic functions. Towards this end, we now summarize how each section of this dissertation relates to these two main results.

The longest section, Chapter 2 is intended as a comprehensive view of the background material underlying our two central results. As such, and given the relatively disjoint nature

[1] A result shown in [5].

of each subsection, we summarize each subsection with its own paragraph. The remaining chapters afterwards will be summarized with their own paragraph.

In Chapter 2.1 we answer the following questions: What is a combinatorial game? What is a Partizan game? And how does one form the surreal numbers using the tools of combinatorial game theory? Further, we provide the foundational recursive constructions for addition, additive inverse, and partial order which are used to endow the class of Partizan games, PG with a partially ordered abelian group structure. In particular, the surreal numbers arise out of the class of partizan games by restricting the partial ordering of Partizan games to a maximum linear ordering of Partizan games.

In Chapter 2.2 we provide an outline of Lurie's proof [14] that PG is a universal embedding structure for partially ordered abelian groups, and relate this proof to the notion of the *s-hierarchy* found in Ehrlich [9, 10, 17–19], and use this to provide constructions of the surreal numbers, in terms of *simplicity*.

In Chapter 2.3, we begin a study of surreal numbers proper, first by defining the Conway normal form of surreal numbers, and motivating the point of view that the surreal numbers form a Hahn field of series. We further summarize work from [7, 20, 21] on nested-truncation rank to motivate the eventual natural definition of a derivative on surreal numbers, along with providing a motivation for a general notion of nested truncation rank when describing genetic functions that need not be simplicity preserving.

In Chapter 2.4, begins by describing gaps in the surreal numbers and how to approach understanding the Dedekind completion of No.[1] In Chapter 2.4.0.1, we prove several elementary results about the intervals satisfying arbitrary Conway cuts. In Chapter 2.4.0.2 we amend results of [3] concerning the Dedekind representation of surreal numbers and gaps, in the process defining two operators to identify Conway cuts with their corresponding minimal realization in the surreal number tree, which we use to define the Dedekind completion. In Chapter 2.4.0.3, we amend the work of [3] to study general On-length sequences of surreal numbers and functions, recovering classical real analysis definitions such as limit and continuity for surreal-valued sequences and functions using Dedekind representations.

In Chapter 2.5, we first summarize various results concerning the sign sequence representation of surreal numbers found in [1, 2], among them several bounds that will be required for subsequent work in this dissertation. This section will be of particular importance since the length of surreal numbers is of central importance for defining our notion of Veblen rank. Additionally, we undertake an extensive examination of the *reduced sign sequences* first discussed in Gonshor, and introduce a strict order relation that is induced by the *reduced sign sequences*. We denote this strict order relation by $\multimap$, and will discuss further properties and consequences of $\multimap$ in Chapter 9.1.

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[1] Unlike the previous three subsections of preliminaries, in this subsection we provide some new definitions and results, although these are ultimately variations or amendments to the work found in [3] and [16].

In Chapter 2.6, we provide the definition for pseudo-absolute values, and prove that several functions on surreal numbers are natural notions of pseudo-absolute values. Importantly, while Gonshor conjectures that the *length function* of a surreal number ι satisfies the following product property:

$$\iota(xy) \leq \iota(x)\iota(y),$$

and consequently will also be a pseudo-absolute value, as noted above, no proof has yet to be published. Later on in Chapter 4, we introduce a weaker notion then *length*, the $\surd$ pseudo-absolute value (referred to as the *surd* pseudo-absolute value), which will suffice to define our notion of Veblen rank.

In Chapter 2.7, we provide several definitions and results from real-algebraic geometry which are of particular importance for defining genetic functions as well as for deriving properties of genetic functions, as the least complicated genetic functions are polynomials with surreal coefficients.

In Chapter 2.8, we provide some background on how one is to study the model theory of a proper class. Following Ehrlich, we motivate the use of NBG, a conservative extension of ZFC. Finally, we summarize several important theorems from Ehrlich which will allow us in future work to conclude that the surreal numbers will be the absolutely homogeneous universal model for certain inductive theories $\mathsf{T}_{\mathcal{G}}$.

In Chapter 3, we begin our study of genetic functions proper. In Chapter 3.1, we first provide several examples of compounds of combinatorial games, and motivate the importance of the *uniformity property* for recursively defined functions on the class of Partizan games. Namely, the uniformity property is what allows for the composition of such functions, which we identify with as game-valued genetic functions. We then proceed to inductively define surreal-valued genetic functions, amending the definition given by [3]. In particular, when defining a new genetic function by adjoining a new function symbol to a given set of genetic functions, when drawing terms for the option sets, we do so now from a ring localized on the cone of strictly positive functions defined in our base ring. Additionally, we impose the *cofinality* condition in order to ensure uniformity. We further provide a multi-variable construction for genetic functions. Afterwards, we discuss the notion of $\leq_s$-minimality in light of our construction of genetic functions and relate this to the ring operations. In Chapter 3.2, we introduce one notion of complexity, generation, which allows for a decomposition of the class of genetic functions in terms of dependence on previously defined genetic functions.

In Chapter 4 we discuss two pseudo-absolute values that could serve as a foundation for studying the complexity of genetic functions. In Chapter 4.1, we study various properties of the α_0 map first discussed in our section on sign sequences. In Chapter 2.6, we build off the results from Chapter 4.1 to introduce the $\surd$ pseudo-absolute value. This pseudo-absolute value is a weaker form of the length function, in that it sends a surreal number to the first term in the Cantor normal form of its length. The bulk of this section consists in proving that $\surd$ is in fact a pseudo-absolute value.

We start Chapter 5 by reviewing a correspondence between tree truncations at heights of specific limit ordinals, and the satisfaction of certain algebraic theories. In Chapter 5.1, we prove our two major Theorems (I) and (II). We first summarize Gonshor's fixed point theorem, and the Veblen hierarchy, showing that the Veblen functions underlying the Veblen hierarchy are proper genetic functions. We then define the notion of partial Veblen rank using the $\surd$-pseudo absolute value discussed in Chapter 2.6, which we use to identify the initial subtree whose elements are of height below a given Veblen function evaluated on a given ordinal that corresponds to a specific ordinal (whence the term partial). From this partial rank, we define Veblen rank as the maximal partial Veblen rank. We then prove that our notion of Veblen rank can be extended to sums, products, and composition of genetic functions, as well as to sets of genetic functions closed under those operations. Since this allows for the notion of Veblen rank to be extended to terms constructed with arithmetic operations and genetic functions, we further extend this notion to formula from a signature consisting of genetic functions. Further, we show that any given genetic function g will be of bounded complexity in the sense that every genetic function g has a Veblen rank $\alpha \in \mathrm{On}$, and so every subtree $\mathrm{No}(\varphi_{\alpha+1}(\gamma))$ will be closed under g, where $\varphi_{\alpha+1}$ corresponds to the $\alpha+1$ Veblen function in the Veblen hierarchy. Further, we identify that each Veblen function φ_α is the canonical example of a genetic function of Veblen rank α. We conclude with Chapter 5.2, where we generalize the notion of nested truncation rank to arbitrary genetic functions. We intend to use this definition in future work in order to interpret genetic functions in Hahn series.

In Chapter 6, we compute the Veblen rank for several genetic functions of interest: among them $\exp, \log, \lambda$ and κ. In Chapter 6.1, we remind the reader of the genetic definitions for ω and ϵ. In Chapter 6.2, we describe the Veblen rank of several classical recursive functions that appear in computability theory. In Chapter 6.3, we compute the Veblen rank for exp and log (along with $\log \circ \omega$, $\mathfrak{g}$ and $\mathfrak{h}$ from [1]). We close with Chapter 6.4 by computing the Veblen ranks of κ and log, and conclude that both have Veblen rank 1, from which we can conclude that while truncating at an epsilon number will mean the corresponding field is closed under finite applications of exp and log, we must truncate at $\varphi_2(\gamma)$ in order to be closed under $\exp - \log$ classes, and log-atomic numbers respectively. This has the implication that ∂_{BM} is closed for truncations at φ_2, but not at ϵ.

In Chapter 7, we take arbitrary sets of genetic functions $\mathcal{G}$, and introduce a definable minimal ordered field closed under the set of genetic functions denoted $Q_{\mathcal{G}}$. We then define $T_{\mathcal{G}}$ to be the corresponding theory of ordered $Q_{\mathcal{G}}$ vector spaces. This Chapter is intended to serve as the basis for future work investigating the properties needed to be satisfied by $\mathcal{G}$ so that $T_{\mathcal{G}}$ will be a homogeneous theory, i.e. one that has both the Joint Embedding Property (JEP) and the Strong Amalgamation property (SAP).

In Chapter 8, we discuss several future directions of research, among them discussions of model complete theories with genetic functions, as well as genetic functions for characteristic p Class-sized fields which are analogoues to the surreal numbers. Notably, Conway discusses the characteristic 2 field in [5], and diMuro [22] provides the corresponding simplicity notion to define a characteristic p field. However, the function theory for these fields remains undeveloped at this point.

In Chapter 9 we provide further details for material appearing in earlier sections. In Chapter 9.1 we study $\multimap$ in further detail, namely providing necessary and sufficient conditions on which an interval of surreal numbers consist of elements whose corresponding sign sequences are all reduced. Finally, we finish the dissertation with a discussion of several known model complete theories satisfied by No in Chapter 9.2, among them real-closed fields, and real-closed fields with exponentiation.

CHAPTER 2

PRELIMINARIES

As summarized at the end of the introduction, we break the preliminaries in this dissertation into eight subsections. The reader is advised to consult the summaries in that section for further guidance. The primary motivations to divide these preliminaries into eight parts are:

1. to give the readers a sufficiently robust background to understand the primary results, theorems (I) and (II) of the introduction;

2. to provide enough context to understand the additional open problems discussed in the concluding remarks and the addendum.

Of all the subsections, the furthest removed from goal (1) are the first two subsections, which pertain to the broader class of combinatorial games, which contains the surreal numbers, and a summary of a proof of the conjecture of Conway that the class of Partizan games is a universal embedding object for partially ordered abelian groups, which is adapted from [14].

While Lurie's proof motivated the content of theorem (II), the primary motivation for including these two subsections has been to give the readers a complete primer on the background material, and to provide context for the discussion of future model theoretic research involving the hierarchical structure of the class of Partizan games and first order algebraic function theories, along with a discussion in the conclusion regarding fuzzy functions, which relies on material from the first subsection.

If readers are comfortable with the definition of surreal numbers as members of a binary tree given in the introduction, they may skip to the third subsection. The third subsection can be skipped if readers are not interested in the several topics mentioned in the conclusion or the addendum, as little use is made of the Conway normal form in the pursuit of theorems (I) and (II); the third subsection is provided for readers interested in understanding the surreal numbers as a real closed valued field, real closed exponential field, or real closed differential field. Subsection 4 and 5 are recommended for all readers interested in the main results. Subsections 6, 7, and 8 are provided primarily to give definitions and technical results used in proving theorems (I) and (II).

2.1 Combinatorial Games

We begin by providing a formal definition for combinatorial games:

Definition 9. *A **combinatorial game** is a two-player game where:*

1. *both players have complete knowledge of the game state at all times;*
2. *and the effects of each move are fully determined beforehand by some ruleset* Γ *that describe how players are to move with respect to their available options, which we define below, given the games' current position.*

We describe such games as containing no hidden information, and no chance elements respectively.

*We use the term **game** to refer to an **individual position** in a combinatorial game. Informally, this may be thought of as the present configuration of game, such as a given board position in a game of chess.*

*Formally, a **ruleset** is a pair* (Γ, N)*, such that* Γ *is set of games, called the **positions** of the ruleset, and* $\mathsf{N} : \Gamma \to \mathbb{N}$*, is a function whose values* $\mathsf{N}(\mathsf{G})$ *is called the **input complexity** of a position* G*.*

Following the literature, we denote the two players as Left and Right. With each position corresponding to a combinatorial game, we say given two positions G *and* H *of a game, that* H *is a **Left option** of* G *if Left can move according to* Γ *directly from the position* G *to position* H*. Similarly, we define a **Right option**.*

We formally represent the current position G *of a game as depending on the options available to the two players by*

$$\mathsf{G} := \mathsf{L}|\mathsf{R},$$

where L *consists of options available to the Left player and* R *consists of the options available to the Right player at the present position.*[1] *Our choice of notation here indicates that for a current position* G *of a combinatorial game, the ruleset* Γ *presents Left with the options of* L *and Right with the options of* R *for the next game position.*

---

[1] *The choice of notation here is to treat* L *and* R *as their own sets, with a dividing line between them, while throughout the rest of the literature, these sets are contained inside set brackets with a dividing line. Since the dividing line can be interpreted in an alternate fashion with standard set-builder notation, we hope this notation is clearer. As an added bonus, when restricting our attention to the surreal numbers, the dividing line is intended to indicate that this is a cut satisfied by an element between the Left and Right option sets.*

A game is ***impartial*** *if both players have the same moves available to them at each sub-position of G. Otherwise, a game is* ***partizan*** *if each player has a distinct move set, and is* ***normal play*** *if the last player to move wins, with the convention that the game is over when at least one of the players has no move available moves. Specifically, a game is over whenever one of the game is in a position where the option set is empty for the player who is currently moving.*

We now introduce four fundamental combinatorial games:

Definition 10. *The simplest combinatorial game is the* ***endgame****. This game is the empty game, where neither Left nor Right have any available options, and so the set of possible sub-positions for both players will be empty. Following the notation above, we denote the endgame by*

$$0 \equiv \{\}\,|\,\{\}\,.$$

It is customary to say that 0 is the endgame precisely because when no moves are available, a game is over. The second player always wins this game under ***normal play****, as the first player will always be unable to make the first move. In* ***misere play****, the first player will always win, as the previous player, in this case the second player, was unable to move. With 0 denoting the endgame, we let*

$$\oplus \equiv 1 \equiv \{0\}\,|\,\{\}$$

$$\ominus \equiv -1 \equiv \{\}\,|\,\{0\}$$

$$* \equiv \{0\}|\{0\}$$

*We use $\oplus$ or 1, to indicate that a game is **positive**, with the meaning that under Normal play, the Left mover wins by playing the endgame option. We use $\ominus$ or -1, to indicate that a game is **negative**, with the meaning that under Normal play, the Right mover wins by playing the endgame option. We use $*$ to denote that a game is **fuzzy**, in that the winner is the first mover, and not necessarily by the options available to the players.*

Remark 4. *In the context of the surreal numbers, positive numbers are precisely the ones where the set of Left options contains 0, and negative numbers are precisely the ones where the set of Right options contains 0. The corresponding game is said to reflect the* strength *of a given Player's position.*

We introduce the following notation to keep track of options:

Notation 1. *For two combinatorial games G, H, H is a Left (respectively Right) **option** of G if Left can move directly from G to H, while H is a **subposition** of G if there exists a sequence of consecutive moves leading from G to H. We indicate a left option of G by G^L and a right option of G by G^R, while the set of left options is denoted by L_G and right options by R_G.*

*We say two games G, H are **identical**, denoted by $\mathsf{G} \equiv \mathsf{H}$, if their respective sets of options agree, i.e. if for every G^L in L_G there is a H^L in L_H such that $\mathsf{G}^\mathsf{L} \equiv \mathsf{H}^\mathsf{L}$, and similarly for the sets of right options.*

This distinction is important, because the discussion of *equality* within Class sized objects in combinatorial game theory is done by restricting our analysis to a definable equivalence relation related to invariance of game outcomes under a *compound operation* (see Chapter 3.1).

Definition 11. *The four **outcome Classes** are:*

1. *First player (the Next player) can force a win, denoted by* $\mathfrak{N}$*;*
2. *Second player (the Previous player) can force a win, denoted by* $\mathfrak{P}$*;*
3. *Left can force a win no matter who moves first, denoted by* $\mathfrak{L}$*;*
4. *Right can force a win no matter who moves first, denoted by* $\mathfrak{R}$*.*

Remark 5. *A quick analysis of* $*$ *shows that the first player to move can force a win.*

Both the Left and Right player have the option to choose $\mathsf{0}$*. That is, for any combinatorial game in the position of* $* = \{\mathsf{0}\} \mid \{\mathsf{0}\}$*, both the Left and Right player can choose to move the game to the position* $\mathsf{0}$*, i.e. to end the game. Specifically, the first player who can move chooses option 0, moving the game from* $*$ *to* $\mathsf{0}$*. Since* $\mathsf{0} = \{\} \mid \{\}$*, the second player to move will then have no available options to choose from, whence the first player wins in conventional play.*

We now provide the first two primitive genetic functions, also called **genetic compounds**:

Definition 12. *We define the **negative** of a game G by*

$$-\mathsf{G} := \left\{ -(\mathsf{G}^{\mathsf{R}}) \right\} \mid \left\{ -(\mathsf{G}^{\mathsf{L}}) \right\}.$$

If G *and* H *are any two combinatorial games, we denote the* ***disjunctive sum*** *of G and H by* $\mathsf{G}+\mathsf{H}$*. When working in a game defined by a disjunctive sum, on a given players turn, a player must move in either G or H, but not both. In larger disjunctive sums, such as*

$$\mathsf{G}_1 + \mathsf{G}_2 + \cdots + \mathsf{G}_n,$$

the current player must move in exactly one component. The genetic *construction of the disjunctive sum is given by*

$$\mathsf{G} + \mathsf{H} := \left\{\mathsf{G}^{\mathsf{L}} + \mathsf{H}, \mathsf{G} + \mathsf{H}^{\mathsf{L}}\right\} \mid \left\{\mathsf{G}^{\mathsf{R}} + \mathsf{H}, \mathsf{G} + \mathsf{H}^{\mathsf{R}}\right\}.$$

Proofs that $+,-$ satisfy the axioms for abelian groups can be found in [5, 23]. Most importantly, $\langle \mathrm{PG}, +, \leq \rangle$ can be regarded as a Class sized partially ordered abelian group, with neutral element the endgame 0. As developed below, one can restrict attention to a proper set of Partizan games by specifying restriction to a given birthday, typically some limit ordinal, in order to preserve closure under $+$.

Definition 13. *Given a combinatorial game with ruleset* (Γ, N) *at position* G*, let* $\mathcal{A}$ *denote the set of all subpositions of* G*. A* ***Left strategy*** *(resp. Right) for* G *is a partial function* $\sigma : \mathcal{A} \to \mathcal{A}$ *such that whenever* $\mathsf{X} \in \mathcal{A}$ *and* X *has at least one Left option, then* $\sigma(\mathsf{X})$ *is defined and is a Left option for* X *(similarly for Right). If* σ *is a strategy for* G*, a* $\sigma-$***play*** *is the sequence of play* $\vec{\mathsf{G}} = \langle \mathsf{G}_i \rangle$ *such that* $\mathsf{G}_{i+1} = \sigma(\mathsf{G}_i)$*. A* ***survival strategy*** *of* G *is a strategy* σ *where the player* *survives every* σ *play. A* ***winning strategy*** *of* G *is a strategy* σ *where according to Normal play, the player can reach the endgame position.*

We recursively define a partial ordering relation on combinatorial games in normal play as follows:

1. *Denote by* $G > 0$ *that there is a winning strategy for Left;*

2. *Denote by* $G < 0$ *that there is a winning strategy for Right;*

3. *Denote by* $G = 0$ *that there is a winning strategy for the second player;*

4. $G \| 0$*, read G is* ***fuzzy*** *if there is a winning strategy for the first player;*

5. $G \geq 0$ *if* $G > 0$ *or* $G = 0$*;* $G \leq 0$ *if* $G < 0$ *or* $G = 0$*;*

6. $G \mid \triangleright 0$ *if* $G > 0$ *or* $G \| 0$*, and* $G \triangleleft \mid 0$ *if* $G < 0$ *or* $G \| 0$.

These are to be read as $G \geq 0$ meaning that supposing Right is the first player, there is a winning strategy for Left, while $G \mid \triangleright 0$ means that there is a winning strategy for Left if Left is the first player [5]. It is worth noting that $G \leq 0$ and the other relations above are well-defined by the **Descending Game Condition** (equivalently, the Conway Induction Principle).[1][2]

For any combinatorial game G, $L_G \geq 0$ denotes that for every $G^L \in L_G$, we have $G^L \geq 0$; similarly, $R_G \leq 0$ if for every $R_G \in G^R$, we have $G^R \leq 0$ (and vice versa for the respective option

[1] The Descending Game Condition is equivalent to the Conway induction principle which states for $n \geq 1$, P is a property of an n-tuple of games $G_1, \ldots, G_n$ if it is a property of all left and right options for G_i, $1 \leq i \leq n$.

[2] Otherwise, without loss of generality, if $G \leq 0$ were not well-defined, then by DGC, some option $G^R \leq 0$ would need to be undefined, and so on, which would violate DGC. See [24] or [5] for further details.

sets and ordering relation). The partial ordering relation for arbitrary combinatorial games G, H can then be restated with the following properties:

1. $\mathsf{G} \geq \mathsf{H}$ if and only if $\neg(\exists \mathsf{G}^{\mathsf{R}} \in \mathsf{R}_{\mathsf{G}} \exists \mathsf{H}^{\mathsf{L}} \in \mathsf{L}_{\mathsf{H}}((\mathsf{G}^{\mathsf{R}} \leq \mathsf{H}) \vee (\mathsf{G} \leq \mathsf{H}^{\mathsf{L}})))$;
2. $\mathsf{G} \leq \mathsf{H} \iff \mathsf{H} \geq \mathsf{G}$;
3. $\mathsf{G} \| \mathsf{H}$ if and only if $\neg(\mathsf{G} \geq \mathsf{H} \vee \mathsf{H} \geq \mathsf{G})$;
4. $\mathsf{G} \mid\triangleright \mathsf{H}$ if and only if $\mathsf{G} \not\leq \mathsf{H}$;
5. $\mathsf{G} \triangleleft\mid \mathsf{H}$ if and only if $\mathsf{G} \not\geq \mathsf{H}$;

Definition 14. *A combinatorial game* G *is **confused** with* H *whenever* $\mathsf{G} \triangleleft\mid\triangleright \mathsf{H}$*. Specifically, a game* G *is **fuzzy** if it is confused with 0.*

Example 1. *Two useful (albeit not-surreal) games are* $\uparrow := \{0\} \mid \{*\}$ *and* $\downarrow := \{*\} \mid \{0\}$*.*

It is immediate from the definitions that $\uparrow > 0$ *since Left can choose to move to 0, while Right can only choose to move to* $*$*, which is losing, since Left can then choose 0.*

Similarly, $\downarrow < 0$*. Moreover,*

$$\cdots > \uparrow + \uparrow > \uparrow > 0 > \downarrow > \downarrow + \downarrow > \cdots .$$

One can show via a straightforward analysis of the game trees that $\uparrow \triangleleft\mid\triangleright *$ *while* $\uparrow + \uparrow > *$*.*

Definition 15. *For games* G, H *belonging to a given (sub)class of combinatorial games, we say* $\mathsf{G} = \mathsf{H}$ *if* $\mathsf{o}(\mathsf{G}+\mathsf{X}) = \mathsf{o}(\mathsf{H}+\mathsf{X})$ *for every game* X *in the given (sub)class of combinatorial games. This construction is equivalent to* $\mathsf{G} = \mathsf{H} \iff \mathsf{G} \geq \mathsf{H} \wedge \mathsf{H} \geq \mathsf{G}$*. The **game value** of* G *is* *its equivalence Class modulo =. A* ***solution*** *to* Γ *is a polynomial-time algorithm for computing* $o(G)$*, the* ***outcome of a game****, for any position* G *of* Γ*.*

Remark 6. *The four outcome Classes of normal play partizan games correspond*[1] *to the following relations to the game* 0*.*

- $G = 0$ *if and only if* $o(G)$ *is that the Second player can always force a win.*
- $G > 0$ *if and only if* $o(G)$ *is that the Left player can always force a win.*
- $G < 0$ *if and only if* $o(G)$ *is that the Right player can always force a win.*
- $G \lhd | \rhd 0$ *if and only if* $o(G)$ *is that the First player can always force a win.*

Proposition 2. *With = given as above,* $*$ *is an order 2 element.*

Proof. A quick analysis of the game $* + *$ shows that the first player will make a losing move to $* + 0 = *$. This establishes the equivalence $* + * = 0$. □

Much of our analysis will depend on substituting one presentation of surreal numbers with another. These substitutions need to be shown to satisfy an equivalence condition of being cofinal/coinitial with a canonical representation. Towards that end, it first needs to be shown that every partizan game G has a **canonical representation** in the following sense:

Definition 16. *Suppose* G *is a partizan game, and* G^{L_1} *and* G^{L_2} *are two Left options, and* G^{R_1} *and* G^{R_2} *are two Right options.*

---

[1] *This result is refered to as the fundamental theorem of combinatorial game theory. Inductive proofs can be found in [5] and in [23].*

- $\mathsf{G}^{\mathsf{L}_1}$ *is* ***dominated by*** $\mathsf{G}^{\mathsf{L}_2}$ *if* $\mathsf{G}^{\mathsf{L}_2} \geq \mathsf{G}^{\mathsf{L}_1}$.
- $\mathsf{G}^{\mathsf{R}_1}$ *is* ***dominated by*** $\mathsf{G}^{\mathsf{R}_2}$ *if* $\mathsf{G}^{\mathsf{R}_1} \geq \mathsf{G}^{\mathsf{R}_2}$.
- $\mathsf{G}^{\mathsf{L}_1}$ *is* ***reversible through*** $\mathsf{G}^{\mathsf{L}_1\mathsf{R}}$, *a Right option of the subposition* $\mathsf{G}^{\mathsf{L}_1}$, *if* $\mathsf{G}^{\mathsf{L}_1\mathsf{R}} \leq \mathsf{G}$.
- $\mathsf{G}^{\mathsf{R}_1}$ *is* ***reversible through*** $\mathsf{G}^{\mathsf{R}_1\mathsf{L}}$, *a Left option of the subposition* $\mathsf{G}^{\mathsf{R}_1}$, *if* $\mathsf{G}^{\mathsf{R}_1\mathsf{L}} \geq \mathsf{G}$.

When options are dominated by or reversible through other options, there is some redundancy. For example, if $\mathsf{G}^{\mathsf{L}_1}$ *is dominated by* $\mathsf{G}^{\mathsf{L}_2}$*, then whenever may be inclined to play from* G *to* $\mathsf{G}^{\mathsf{L}_1}$*, Left would do at least as well to play from* G *to* $\mathsf{G}^{\mathsf{L}_2}$ *instead. On the other hand, for any Left option that is reversible through a Right option* $\mathsf{G}^{\mathsf{L}_1\mathsf{R}}$*, we can remove* $\mathsf{G}^{\mathsf{L}_1}$ *from* G *by replacing it with all the Left options of the sub-position* $(\mathsf{G}^{\mathsf{L}_1\mathsf{R}})$*. This operation is called* ***bypassing the reversible option*** *A game* G *is in* ***canonical form*** *if no subposition of* G *has any dominated or reversible options. A game* G *is* ***position closed*** *if for every* $\mathsf{A} \in \mathsf{L}_\mathsf{G}$, $\mathsf{L}_\mathsf{A} \subset \mathsf{L}_\mathsf{G}$ *and for every* $\mathsf{B} \in \mathsf{R}_\mathsf{G}$, $\mathsf{R}_\mathsf{B} \subset \mathsf{R}_\mathsf{G}$.

The Class of partizan games, $\widetilde{\mathrm{PG}}$, is inductively formed as:

$$\widetilde{\mathrm{PG}} := \bigcup_{\alpha \in \mathrm{On}} \tilde{\mathbb{G}}_\alpha$$

where for each α,

$$\tilde{\mathbb{G}}_\alpha := \left\{ \mathsf{L}_\mathsf{G} \mid \mathsf{R}_\mathsf{G} \colon \mathsf{L}_\mathsf{G}, \mathsf{R}_\mathsf{G} \subset \bigcup_{\beta \in \alpha} \tilde{\mathbb{G}}_\beta \right\}.$$

The **formal birthday** of a game G is the least ordinal such that $\mathsf{G} \in \tilde{\mathbb{G}}_\alpha$.

Remark 7. *As mentioned in the introduction, birthdays and simplicity are related, and whenever talking strictly about surreal numbers or surreal valued functions, we prefer working explicitly with respect to the concept of simplicity, which allows us to directly invoke a tree structure, as opposed to the substantially more complicated poset structure Partizan games.*

Notation 2. *We let* $\mathbb{G}_\alpha$ *denote the set of values of games in* $\tilde{\mathbb{G}}_\alpha$*. In particular, and let* PG *denote the Class of games values of* $\tilde{\mathrm{PG}}$ *- care needs to be taken when constructing this equivalence relation, since* $\tilde{\mathrm{PG}}$ *is a proper Class, and the equivalence Class of any particular game is in general a proper Class, so the formation of a Class of all such Classes is not legally formable inside. The workaround in practice has been to work in a conservative extension of set theory, such as* NBG *(see [9, 17]), and to restrict each =-equivalence Class to the elements of minimal set-theoretic rank.*

Definition 17. *Alternatively, we can construct the Class of transfinite partizan games as follows:*

Let L, R *denote two sets of games in* PG*. Then the ordered pair* $\mathsf{G} := \langle \mathsf{L}, \mathsf{R} \rangle$ *is a game provided that* G *also satisfies the* ***descending game condition****: there is no infinite sequence of games* $\mathsf{G}^i = \mathsf{L}^i | \mathsf{R}^i$ *such that* $\mathsf{G}^{i+1} \in \mathsf{L}^i \cup \mathsf{R}^i$ *for all* $i \in \omega$*.*

2.2 Universal Embedding

We can endow the Class $\langle \mathrm{PG}; +, -, \leq \rangle$ with the structure of a partially ordered Abelian group, as the rest of Chapter 7 in [5] proceeds to show. Furthermore, PG is the universally embedding partially ordered abelian group in the sense that every partially ordered abelian group is isomorphic to some subgroup of PG, as will be summarized below (see [14] for further details). Precisely,

Theorem 4. *Suppose that* $S \subseteq S'$ *are two sets such that* S, S' *are partially ordered abelian groups, and* $\phi : S \to \mathrm{PG}$ *is an order-preserving homomorphism. Then there exists an order-preserving homomorphism* $\phi' : S' \to \mathrm{PG}$ *such that* $\phi'|_S = \phi$.

It is instructive to summarize Lurie's proof of Theorem 4. To begin, Lurie proved a *weak embedding theorem* where S, S' instead are partially ordered sets. This tracks with the approach employed in [9, 10, 17–19] where the author studies structures which are initial subtrees in the s-hierarchy, which can be seen as a restriction of the implicit *hereditary hierarchy* or *h-hierarchy* employed by Lurie in [14].

Definition 18. *Let* $S \subsetneq \mathrm{PG}$ *be a proper set of games.* S *is* ***hereditary*** *if for every* $G \in S$, $L_G \subseteq S$ *and* $R_G \subseteq S$. *In particular, for any* $S \subseteq \mathrm{PG}$ *that is a proper set, we can always form a hereditary set* $S^* \supseteq S$ *by recursively adjoining all options of elements of* S, *and the options of those elements, and so on.*

Lurie uses the following lemma to guarantee the existence of a game G that allows for the extension of an order preserving homomorphism.

Lemma 1. *Suppose* $S \subsetneq \mathrm{PG}$ *is a set of games such that* $S = S^*$, *and let* $L, R \subset S$ *such that*

1. L *is closed downwards in* S, *i.e. if* $y \leq x$ *and* $x \in L$, *then* $y \in L$;
2. R *is closed upwards in* S, *i.e. if* $x \leq y$ *and* $x \in R$, *then* $y \in R$;

3. $L \le R$*, i.e. for all* $x \in L$ *and* $y \in R$*,* $x \le y$*.*

Now let $G = L_G | R_G$ *be a game with* $L_G := S \backslash R$ *and* $R_G := S \backslash L$*. Then for any* $x \in S$*,* $G \le x$ *if and only if* $x \in R$*, and* $x \le G$ *if and only if* $x \in L$*.*

In order to prove the weak embedding theorem, one uses Zorn's lemma to reduce the problem of extending an embedding in general to extending the definition of the map by a single new element, as in Ehrlich [9, 17], after one enlarges a partially ordered set S into a hereditary set.

For the strong embedding result, Lurie requires that a partially ordered set S be enlarged so that it is both hereditary, but also enlarged in a way that accounts for the arithmetic of games. This enlargement is **justified** in the following sense:

Definition 19. *Let* S *be a subgroup of* PG*. A* ***framing*** *of* S *is a collection of subsets* $S_i \subseteq S$*, indexed by the integers, such that*

- $S_i + S_j \subseteq S_{i+j}$*;*
- $g \in S_0$ *if and only if* $g \ge 0$*.*

If $S \subseteq S'$ *is a subgroup of* PG*, then a framing of* S' ***extends*** *a framing of* S *if* $S_i = S'_i \cap S$ *for all* $i \in \mathbb{Z}$*. For a framed subgroup* S *of* PG*, with* $g \notin S_n$ *for some* $g \in S$ *and* $n > 1$*, the pair* (g, n) *is* ***justified*** *if there exists an* $x \in S_{-1}$ *such that* $g + x \notin S_{n-1}$*. Similarly if* $g \notin S_{-n}$*, we say* $(g, -n)$ *is justified if there exists* $x \in S_1$ *such that* $g + x \notin S_{-n+1}$*. Any framed subgroup* S *is called* ***justified*** *if whenever* $g \notin S_n$*,* $n \ne -1, 0, 1$*, then* (g, n) *is justified in* S*.*

Lurie shows that for any framed subgroup S of PG such that $g \notin S_n$ and $g \in S$, there is a framed subgroup of PG extending S in which (g, n) is justified. Having proved this, Lurie proceeds to show that any framed subgroup of PG can be extended to a justified, hereditarily framed subgroup. This entails that for any framed subgroup S, there exists a game x such that $S_n = \{y \in S \mid nx \leq y\}$ for every $n \in \mathbb{Z}$, which is necessary to prove Theorem 4. Importantly, these operations are defined with respect to arbitrary combinatorial games, although our primary interest will be the restriction of these operations solely to the surreal numbers.

We define surreal numbers as a subclass of PG as follows:

Definition 20. *Inductively, we can define numbers as games* $G = L_G|R_G$ *whose left and right options are sets of numbers such that* $L_G < R_G$*. Precisely, we let* No *be the subclass of* PG *defined by the following* ***simplicity rule****: if all options* G^L, G^R *for a game G are numbers such that* $G^L < G^R$*, then* $G = L_G|R_G$ *is the simplest number such that* $L_G < G < R_G$*.*

This definition of simplicity can be restated by redefining PG and No as follows:

Definition 21. *With* $\mathrm{PG}_0 = \{0\}$*, let*

$$\mathrm{PG}_\alpha = \{\left\{G^L\right\} \mid \left\{G^R\right\} \mid L_G, R_G \subset \bigcup_{\beta \in \alpha} \mathrm{PG}_\beta\},$$

and set

$$\mathrm{PG} := \bigcup_{\alpha \in \mathrm{On}} \mathrm{PG}_\alpha$$

Furthermore, letting $\mathrm{No}_0 = \mathrm{PG}_0$*, let*

$$\mathrm{No}_\alpha = \{\left\{a^L\right\} \mid \left\{a^R\right\} : L_a < R_a \wedge L_a, R_a \subset \bigcup_{\beta \in \alpha} \mathrm{No}_\beta\}$$

and set

$$\mathrm{No} = \bigcup_{\alpha \in \mathrm{On}} \mathrm{No}_\alpha$$

For a given game (number) G*, we say the* ***birthday****, (or preferably* ***length****) of* G *is the least* α *such that* $G \in \mathrm{PG}_\alpha$*, (respectively,* $G \in \mathrm{No}_\alpha$*). We say* x *is* ***simpler than*** *y if the least* α *such that* $x \in \mathrm{No}_\alpha$ *is less than the least* β *such that* $y \in \mathrm{No}_\beta$*. Moreover, for each* $G \in \mathrm{No}_\alpha \setminus \bigcup_{\beta \in \alpha} \mathrm{No}_\beta$*, the positions defining* G *are known as the canonical representation, i.e.*

$$G^L < G < G^R$$

such that G^L, G^R *are simpler than* G*. Finally, the order relation involved in the definition above is simultaneously defined by induction at each stage* α *according to the definition given above.*

In turn, we have the following definitions recapitulating the idea that games can be understood in terms of numbers, courtesy of [5, 16, 23]:

Definition 22. *Suppose* $x \in \mathrm{PG}$*. We say* x *is a* ***number*** *if* $y^L < y^R$ *for every subposition y of x, and every* y^L *and* y^R*. An* ***(omnific) integer*** *is any number* a *such that* $a = \{a-1\}|\{a+1\}$*. We denote the Class of numbers and omnific integers by* No *and* Oz *respectively.*[1]

Finally, let $\mathcal{I} \subset \mathrm{PG}$*. We say* $\mathcal{I}$ *is an* ***interval*** *if whenever* $x, y \in \mathcal{I}$ *such that* $y \leq x$*, and* $y \leq z \leq x$*, then* $z \in \mathcal{I}$*.*

---

[1] Oz *is of great interest, as* Oz *is a maximal integral domain such that* $\mathsf{Frac}(\mathrm{Oz}) = \mathrm{No}$*, see [5] for further details.*

2.3 Normal Forms and Standard Form

Every surreal number $x \in \mathrm{No}$ has a **Conway normal form**, which can be regarded as a formal sum consisting of real coefficients and monomials $\mathfrak{m}$ drawn from a Class of monomials $\mathfrak{M}$. Specifically, for every $x \in \mathrm{No}$, we set $\nu a \in \mathrm{On}$, a sequence of real numbers $(r_\mu)_{\mu \in \nu a}$, and a descending sequence of surreal numbers $(a_\mu)_{\mu \in \nu a}$ such that

$$x = \sum_{i \in \alpha} \omega(a_\mu) r_\mu \text{ where } (a_\mu)_{\mu \in \nu a} \downarrow$$

We define **support** of a surreal number a to be the set

$$S(a) := \{y \in \mathrm{No} \mid \exists \beta \in \alpha (y = y_\alpha \wedge r_\alpha \neq 0)\}.$$

Definition 23. *Let $x \in \mathrm{No}$. If we express $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}} \mathfrak{m}$, as above:*

1. *The **support** of x is the set $S(x) := \{\mathfrak{m} \in \mathfrak{M} \mid x_{\mathfrak{m}} \neq 0\}$;*
2. *The **terms** of x are the elements of the set $\{x_{\mathfrak{m}} \mathfrak{m} \mid x_{\mathfrak{m}} \neq 0\} \subset \mathbb{R}^{\times} \mathfrak{M}$;*
3. *The **coefficient** of $\mathfrak{m}$ in x is $x_{\mathfrak{m}}$;*
4. *The **leading monomial** of x is the maximal monomial in $S(x)$;*
5. *The **leading term** of x is the leading monomial multiplied by its coefficient.*
6. *Given $\mathfrak{m} \in \mathfrak{M}$, the **truncation** of x at $\mathfrak{m}$ is the number*

$$x \upharpoonright \mathfrak{m} := \sum_{\mathfrak{m} < \mathfrak{n}} x_{\mathfrak{n}} \mathfrak{n}$$

.

7. *If* $y \in \mathrm{No}$ *is a truncation of* x*, we denote this by* $y \trianglelefteq x$*.*

The above definitions are intended to track with those we establish for Hahn fields. As an aside, we introduce the following important properties for distinguished subfields of a Hahn field $\mathbb{R}((t^{\Gamma}))$ with value group Γ.

Definition 24. *Let* Γ *be an ordered class. We let* $\mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$ *denote the ordered* ***Hahn group*** *of formal power series consisting of all formal power series of the form* $\sum_{\alpha\in\beta} t^{s_\alpha} r_\alpha$*, where* $(s_\alpha)_{\alpha\in\beta}$ *is a possibly empty descending sequence of eleemnts of an ordered class* S*, and* $r_\alpha \in \mathbb{R}^{\times}$ *for each* α*. We call* Γ *the* ***value class*** *of* $\mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$*. If* Γ *is an ordered abelian group, then* Γ *is the* ***value group****, and* $\mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$ *is a* ***Hahn field****.*

For subclasses $S \subset \Gamma$*, let* t^S *denote* $\{t^s : s \in S\}$*. A subgroup (subfield)* $G \subset \mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$ *is* ***cross-sectional*** *if* $\{t^g : g \in \Gamma\} \subset G$*.*

For an element $x \in \mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$*, where* $x = \sum_{\alpha\in\beta} t^{y_\alpha} r_\alpha$*, we say* y *is a* ***truncation*** *of* x *if* $y = \sum_{\alpha\in\gamma} t^{y_\alpha} r_\alpha$ *with* $\gamma \leq \beta$*.*

A subgroup (subfield) $G \subset \mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$ *is* ***truncation closed*** *if every truncation of every member of* G *is a member of* G*.*

We now cite Theorem 1 from [25], which will be of great importance for our subsequent model theoretic work.

Theorem 5. *An ordered abelian group is isomorphic to an initial subgropu of* No *if and only if it is isomorphic to a truncation closed, cross-sectional subgroup* G *of a power series group* $\mathbb{R}((t^{\Gamma}))_{\mathrm{On}}$*, where*

1. Γ *is isomorphic to an initial ordered subclass of* No*;*
2. *every y-coefficient group* $\mathbb{R}_y = \{r \in \mathbb{R} \colon rt^y \in G\}$ *of* G *is an initial subgroup of* $\mathbb{R}$*;*
3. $\mathbb{D} \subseteq \mathbb{R}_y$ *for all* $x, y \in \Gamma$ *where* $y \in R_x$.

Following [11], we have the following inductively defined map $\Sigma : \mathbb{R}((\mathfrak{M})) \to \mathrm{No}$ where $\mathbb{R}((\mathfrak{M}))$ can be substituted with $\mathbb{R}((t^{\mathrm{No}}))_{\mathrm{On}}$:

Definition 25. *Recall given a field* K *and a set (or Class) of monomials* $\mathfrak{M}$*, we let* $K((\mathfrak{M}))$ *denote the field of* formal Laurent series*. Let* $f \in \mathbb{R}((\mathfrak{M}))$*. With* $f_{\mathfrak{m}} \in \mathbb{R}$ *and* $\mathfrak{m} \in \mathfrak{M}$*, and* $f \upharpoonright \mathfrak{m}$ *denoting the truncation at* $\mathfrak{m}$*, we define:*

1. *If* $S(f) = \emptyset$*, then* $\Sigma f := 0 \in \mathrm{No}$*;*
2. *If* $S(f)$ *contains a smallest monomial* $\mathfrak{n}$*, define*

$$\Sigma f := \Sigma f \upharpoonright \mathfrak{n} + f_{\mathfrak{n}}\mathfrak{n}$$

3. *If* $S(f) \neq \emptyset$ *and has no smallest monomial, with* q^L, q^R *arbitrary dyadic rationals such that*

$$q^L < f_{\mathfrak{m}} < q^R$$

then

$$\Sigma f := \left\{ \Sigma f \restriction \mathfrak{m} + q^L \mathfrak{m} \right\} | \left\{ \Sigma f \restriction \mathfrak{m} + q^R \mathfrak{m} \right\}$$

In particular, we recognize that

$$\mathrm{No} = \mathbb{R}((\omega^{\mathrm{No}})) = \mathbb{R}((\mathfrak{M}))$$

Given that we may regard ω^{No} as the complete system of representatives of Archimedean equivalence classes of $\mathrm{No}_{>0}$, and that we can take the map Ind from [1] that sends each surreal number to the exponent a_0 of its leading monomial in normal form, we may regard the surreal numbers as a valued field which is its own value group. Moreover, Ind can be identified with the natural Krull valuation ℓ of the real closed field No, with $\mathrm{Ind}(\omega^a) = a$ for all $a \in \mathrm{No}$. We can further deduce that No is a **Hahn field of series** in the following sense [26]:

Theorem 6. 1. *For any* $a \in \mathrm{No}$, $(\omega(\omega(a))$ *is the representative of minimal length in* $\mathrm{No}^{>0}_{>0}$, *i.e.*

$$\forall x, y \in \mathrm{No}^{>0}_{>0}, x \sim_{\mathrm{comp}} y \iff (\exists n \in \omega(x^n \geq y \geq x^{1/n})$$

We set $x \sim_{\mathrm{comp}} \frac{1}{x}$.

2. *Any* $a \in \mathrm{No}$ *can be uniquely written as*

$$a = \sum_{i \in \lambda} \left(\prod_{j \in \lambda_i} \left(\omega^{\omega(b_{i,j})} \right)^{s_{i,j}} \right) r_i$$

where for any i,

$$a_i = \sum_{j \in \lambda_i} \omega^{b_{i,j}},$$

the $(a_i)_{i\in\lambda}, (b_{i,j})_{j\in\lambda_i}$ *form descending sequences of surreals, and for any* i, j*, we have* $s_{i,j}, r_i \in \mathbb{R}^\times$*, so that*

$$\omega(a_i) = \prod_{j<\lambda_i} \left(\omega^{\omega(b_{i,j})}\right)^{s_{i,j}}$$

In particular, using the definition of Hahn fields from [27], we find

$$\mathrm{No} = \mathbb{R}\left(\left(\left(\omega^{\omega^{\mathrm{No}}}\right)^{\mathbb{R}}\right)\right)$$

We let $\mathbb{J} \subset \mathrm{No}$ denote the (class) non-unital ring of infinite surreal numbers. Specifically, they're the surreal numbers whose supports have infinite monomials, so

$$\mathbb{J} := \{a \mid \forall y \in S(a) \exists z > 0 (y = \omega(z))\} = \mathrm{No}^{>0} \cup \{0\}$$

It follows from the constructions above that

$$\omega(\mathrm{No}) = \exp(\mathbb{J})$$

A very important substructure of the surreal numbers are the **omnific integers**.

Definition 26. *The **omnific integers** are the numbers of the form* $x = \{x-1\}|\{x+1\}$*. The class of omnific integers, denoted* Oz*, has the direct sum decomposition* $\mathbb{J} \oplus \mathbb{Z}$*.*

Remark 8. *We note that surreal numbers can be given a natural direct sum decomposition of*

$$\mathrm{No} = \mathbb{J} \oplus \mathbb{R} \oplus o(1)$$

where $o(1)$ *denotes the class of infinitesimal numbers.*

As an alternative to Conway normal form where the monomials are based ω, we may put the surreal numbers in **Ressayre normal form**, where the monomials are based exp, as in for each $a \in \mathrm{No}$ there is an ordinal $\rho a \in \mathrm{On}$ and a descending sequence y_μ such that

$$a = \sum_{\mu \in \rho a} \exp(y_\mu) r_\mu$$

where $(y_\mu)_{\mu \in \rho a}$ is a descending sequence of surreal numbers.

Definition 27. *Given* $x \neq 0$ *with Ressayre normal form* $\sum_{a \in \mathbb{J}} r_a \exp(a)$*, with* $r_a \neq 0$ *if and only if* $a \in S(x)$*, we define* $\ell : \mathrm{No}^\times \to \mathbb{J}$ *by* $\ell(x) = \max\{a \in \mathbb{J} \mid r_a \neq 0\}$*.*

Remark 9. *The map above can be regarded as the logarithm* a *of the largest monomial* $\mathfrak{m} = \exp(a)$ *appearing in the Conway normal form of x. Further,* $-\ell$ *defines a Krull valuation on* No*, given that*

1. $\ell(x + y) \leq \max\{\ell(x), \ell(y)\}$
2. $\ell(xy) = \ell(x) + \ell(y)$.

An almost immediate consequence of these two normal forms is that the surreal numbers can be understood as a valued field which is its own valued group.

The following are facts about the normal form with respect to the simplicity hierarchy (see [9] for more details):

Fact 1. *1. For all* $a, b \in \mathrm{No}$, $\omega(a) <_s \omega(b)$ *if and only if* $a <_s b$.

2. *If* $a <_s b$, *then* $\omega(x)a <_s \omega(x)b$.

3. $\sum_{i\in\mu} \omega(y_i)r_i <_s \sum_{j\in\nu} \omega(y_j)r_j$ *whenever* $\mu <_s \nu$

4. *If* μ *is a limit ordinal, then with* γ *ranging over* μ, *and* n *ranging over* ω, *we have*

$$\sum_{i\in\mu} := \left\{ \sum_{i\in\gamma} \omega(y_i)r_i + \omega(y_\gamma)(r_\gamma - \frac{1}{2^n}) \right\} \mid \left\{ \sum_{i\in\gamma} \omega(y_i)r_i + \omega(y_\gamma)(r_\gamma + \frac{1}{2^n}) \right\}$$

5. *If* $r \in \mathbb{R}\backslash\mathbb{D}$, *or* $r \in \mathbb{D}\backslash\mathbb{Z}$ *and there is no* y^L, *then*

$$\omega(y)r = \left\{\omega(y)r^L\right\} \mid \left\{\omega(y)r^R\right\}$$

6. *For all* r^L, r^R,

$$\omega(y)r^L <_s \omega(y)r$$

and

$$\omega(y)r^R <_s \omega(y)r$$

7. *If* $r \in \mathbb{D}\backslash\mathbb{Z}$, *and there exist* y^L, *then*

$$\omega(y)r = \left\{\omega(y) + r^L + \omega(y^L)n\right\} \mid \left\{\omega(y)r^R - \omega(y^L)n\right\}$$

8. $\omega(y)r^L + \omega(y^L)n <_s \omega(y)r$.

9. $\omega(y)r^R - \omega(y^L)n < s\omega(y)r$.

10. *For all* n, $\frac{1}{2^n}\omega(x^R) <_s \omega(x)$.

Consequently, we have

Proposition 3. *If* $x \trianglelefteq y$, *then* $x <_s y$.

We can also see that $\trianglelefteq$ is a weakening of $<_s$ once we have our results on the sign sequence of surreal numbers.

2.3.0.1 Summability

Having identified surreal numbers with $\mathbb{R}((\mathfrak{M}))$, we can explore the notion of infinite sums. Namely,

Definition 28. *Let* $(x_i)_{i\in I}$ *be an indexed set of surreal numbers. We say* $(x_i)_I$ *is* ***summable*** *if* $\bigcup_I S(x_i)$ *is reverse well-ordered, and if for each* $\mathfrak{m} \in \bigcup S(x_i)$ *there are only finitely many* $i \in I$ *such that* $\mathfrak{m} \in S(x_i)$.

When $(x_i)_I$ *is summable, then the* ***sum***

$$y := \sum_{i\in I} x_i$$

is the unique surreal number such that:

- $S(y) \subseteq \bigcup_I S(x_i)$

- *for every* $\mathfrak{m} \in \mathfrak{M}$, $y_{\mathfrak{m}} = (\sum_{i \in I} x_i)_{\mathfrak{m}} = \sum_{i \in I} x_{i\mathfrak{m}}$

Definition 29. *A function* $F : \mathrm{No} \to \mathrm{No}$ *is* ***strongly linear*** *if for all* $x = \sum x_{\mathfrak{m}} \mathfrak{m}$,

$$F(x) = \sum x_{\mathfrak{m}} F(\mathfrak{m}).$$

In particular, $(x_{\mathfrak{m}} F(\mathfrak{m}))$ *is summable.*

Proposition 4. *IF* F *is a strongly linear function, then for any summable* (x_i)*, the family* $(F(x_i))$ *is summable and*

$$F(\sum x_i) = \sum F(x_i)$$

Proof. The following one line proof is from [11]:

$$F(\sum x_i) = F\left(\sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{i \in I} x_i\right)_{\mathfrak{m}} \mathfrak{m}\right) = \sum_{\mathfrak{m} \in \mathfrak{M}} \sum_{i \in I} x_{i\mathfrak{m}} F(\mathfrak{m}) = \sum_{i \in I} F(x_i)$$

□

2.3.0.2 Nested truncation and standard forms

As observed above, we have that the ω map monotonically preserves simplicity, but that the exp map does not (as will be made clearer once we have log defined). However, there are a subclass of numbers where exp is a monotonic map preserving simplicity, as the following theorem from [11] shows

Theorem 7. *If* $\mathfrak{a}, \mathfrak{b} \in \mathbb{J}$ *and* $\mathfrak{a} \trianglelefteq \mathfrak{b}$*, then* $\exp(\mathfrak{a}) \leq_s \exp(\mathfrak{b})$.

In general, the above result is not sufficient for studying exp and $\leq_s$. The authors [11] remedied this by introducing the notion of **nested truncation** and a corresponding rank.

Definition 30. *A finite sum of surreal numbers* $y = x_1 + x_2 + \cdots + x_n$ *is in* ***standard form*** *if* $S(x_1) > S(x_2) > \cdots > S(x_n)$.

For $x \in \mathrm{No}^\times$, *set* $sgn(x) = 1$ *if* $x > 0$ *and* $sgn(x) = -1$ *otherwise.*

We then inductively define ranks $\trianglelefteq_n$ *on* $\mathrm{No}^\times$ *over* $n \in \omega$ *as follows:*

1. $x \trianglelefteq_0 y$ *if* $x \trianglelefteq y$;
2. $x \trianglelefteq_{n+1} y$ *if there are* $a \trianglelefteq_n b$ *with* $a, b \in \mathbb{J}^*$, *and* $z, w \in \mathrm{No}$ *and* $r \in \mathbb{R}^\times$ *such that*

$$x = z + sgn(r)\exp(a)$$

$$y = z + r\exp(b) + w$$

where both sums are in standard form.

We say $x \trianglelefteq y$, *or that* x *is a* ***nested truncation of y*** *if there is an* n *such that* $x \trianglelefteq_n y$.

$\trianglelefteq$ induces a foundation rank, which we define as follows:

Definition 31. *For all* $x \in \mathrm{No}^\times$, *the* ***nested truncation rank***, $\mathrm{NR}(x)$ *is defined by*

$$\mathrm{NR}(x) := \sup\{\mathrm{NR}(y) + 1 \mid y \trianglelefteq x\}$$

With $\mathrm{NR}(0) = 0$

Remark 10. *Since all real numbers have no proper truncations, we find that $\mathbb{R}$ has nested truncation rank 0*

Theorem 8. $\trianglelefteq$ *partially orders* $\mathrm{No}^{\times}$.

Proof. It is immediate that $\trianglelefteq$ is reflexive since $x \trianglelefteq x$ for all x.

We prove antisymmetry as follows. Suppose for some n that $x \trianglelefteq_n y$ and $y \trianglelefteq x$. Immediately o.t. S(x) = o.t. S(y). Proceeding by induction on n, for n=0, we have $x = y$. For $n > 0$, write x and y in the standard forms as above with $a \trianglelefteq_{n-1} b$. By the observation on order types, we have that $w = 0$ and by the hypothesis that $y \trianglelefteq x$, we have that $r = \mathrm{sgn}(r)$, and $b \trianglelefteq a$. But then by our inductive hypothesis, we have $a = b$, and hence $x = y$.

We prove transitivity as follows: Supposing for $n, m \in \omega$ that $x \trianglelefteq_n y \trianglelefteq_m z$. If $n = 0$, then $x \trianglelefteq y$, from which it follows that $x \trianglelefteq_m z$ as a truncation. Similarly, if $m = 0$, we have $y \trianglelefteq z$, from which $x \trianglelefteq_n z$. If $m, n > 0$, write y and z in the following standard forms

$$y = u + \mathrm{sgn}(r)\exp(b)$$

$$z = u + r\exp(c) + w$$

with $b \blacktriangleleft_{m-1} c$ (and $b, c \in \mathbb{J}^*$). We are done if $x \trianglelefteq_n z$, as $z \trianglelefteq u$ implies $x \trianglelefteq_n u$.

Otherwise, $\neg(x \trianglelefteq_n z)$, so we must have $x = z + \mathrm{sgn}(r)\exp(a)$ with $a \trianglelefteq_{n-1} b$ and $a \in \mathbb{J}^*$. By our induction hypothesis, we have that $a \trianglelefteq c$, from which $x \trianglelefteq u$. □

Berarducci and Mantova [11] establish the following facts on $\trianglelefteq$:

Fact 2. *1. For all* $\mathfrak{x}, \mathfrak{y} \in \mathbf{No}^{\times}$ *and* $z \in \mathbf{No}$*, if* $z + \mathfrak{x}$ *and* $z + \mathfrak{y}$ *are in standard form, then* $\mathfrak{x} \trianglelefteq\!\!\!\blacktriangleleft \mathfrak{y} \iff \mathfrak{x} + z \trianglelefteq\!\!\!\blacktriangleleft \mathfrak{y} + z$.

2. *For all* $\mathfrak{x} \in \mathbf{No}^{\times}$*, and* $\mathfrak{m} \in \mathfrak{M}$*, if* $\mathfrak{x} \underline{\blacktriangleleft} \mathfrak{m}$*, then* $\mathfrak{x} \in \mathfrak{M}$.

3. $\underline{\blacktriangleleft}$ *is the smallest transitive relation such that:*

   - *for all* $\mathfrak{x}, \mathfrak{y} \in \mathbf{No}^{\times}$*,* $\mathfrak{x} \trianglelefteq \mathfrak{y} \Rightarrow \mathfrak{x} \underline{\blacktriangleleft} \mathfrak{y}$*;*
   - *for all* $\mathfrak{a}, \mathfrak{b} \in \mathbb{J}^*, \mathfrak{a} \underline{\blacktriangleleft} \mathfrak{b}$ *implies that* $\exp(\mathfrak{a}) \underline{\blacktriangleleft} \exp(\mathfrak{b})$ *and* $-\exp(\mathfrak{a}) \underline{\blacktriangleleft} -\exp(\mathfrak{b})$*;*
   - *for all* $\mathfrak{m} \in \mathfrak{M}^{\neq 1}$ *and* $\mathfrak{r} \in \mathbb{R}^{\times}$*, sgn*$(\mathfrak{r})\mathfrak{m} \underline{\blacktriangleleft} \mathfrak{r}\mathfrak{m}$*;*
   - $\forall \mathfrak{x}, \mathfrak{y} \in \mathbf{No}^{\times}$*, and* $z \in \mathbf{No}$*, if* $z + \mathfrak{x}, z + \mathfrak{y}$ *are both in standard form, then if* $\mathfrak{x} \underline{\blacktriangleleft} \mathfrak{y}$*, then* $z + \mathfrak{x} \underline{\blacktriangleleft} z + \mathfrak{y}$.

4. *For all* $\mathfrak{x} \in \mathbf{No}$*, the class* $\{\mathfrak{y} \in \mathbf{No} \mid \mathfrak{x} \trianglelefteq \mathfrak{y}\}$ *is convex.*

5. *For all* $\mathfrak{x} \in \mathbf{No}^{\times}$*, the class* $\{\mathfrak{y} \in \mathbf{No}^{\times} \mid \mathfrak{x} \underline{\blacktriangleleft} \mathfrak{y}\}$ *is convex.*

6. $\mathbb{J}$ *is closed under* $\trianglelefteq$ *and* $\underline{\blacktriangleleft}$*. Namely, for all* $\mathfrak{x} \in \mathbf{No}^{\times}$ *and* $\mathfrak{a} \in \mathbb{J}^*$*, if* $\mathfrak{x} \underline{\blacktriangleleft} \mathfrak{a}$*, then* $\mathfrak{a} \in \mathbb{J}^*$.

7. $\mathfrak{x} \underline{\blacktriangleleft} \mathfrak{y}$ *implies that* $\mathfrak{x} \leq_s \mathfrak{y}$*, implying that* $\underline{\blacktriangleleft}$ *is well-founded, so that* $\underline{\blacktriangleleft}$ *has an associated ordinal rank which we'll call our* ***nested tree rank****.*

We state without proof several facts regarding the nested truncation rank:

Fact 3. *1. For all* $\mathfrak{x} \in \mathbf{No}$*,* $\mathsf{NR}(\mathfrak{x}) = \mathsf{NR}(-\mathfrak{x})$.

2. *For all* $\mathfrak{a} \in \mathbb{J}$*,* $\mathsf{NR}(\mathfrak{a}) = \mathsf{NR}(\pm\exp(\mathfrak{a}))$.

3. *For all* $\mathfrak{m} \in \mathfrak{M}^{\neq 1}$*, and* $\mathfrak{r} \in \mathbb{R}^{\times}$*, if* $\mathfrak{r} \neq \pm 1$*, then* $\mathsf{NR}(\mathfrak{r}\mathfrak{m}) = \mathsf{NR}(\mathfrak{m}) + 1 > \mathsf{NR}(\mathfrak{m})$.

4. *If* $x \neq 0$*, and if* $r\mathfrak{m}$ *is a term of* x*, then* $\mathsf{NR}(r\mathfrak{m}) \leq \mathsf{NR}(x)$ *and if* $\mathsf{NR}(\mathfrak{m})$ *is not minimal in* $S(x)$*, then* $\mathsf{NR}(r\mathfrak{m}) < \mathsf{NR}(x)$.

2.4 Fundamentals of Surreal Analysis

The primary goal of this subsection is to reconcile the existing approaches to the analysis of surreal numbers and surreal-valued functions that exist in the literature (specifically [16], [8], and [3]) with the simplicity structure of the surreal binary tree that we will use to define surreal-valued genetic functions in Chapter 55. In particular, our definition of genetic functions extends our identification with simplicity as the minimal set theoretic realization of a cut defined by Left and Right option sets (a perspective developed in detail below).

Intuitively, our goal means that we want to be able to reason about the limits of sequences of surreal numbers in a fashion analogous to real analysis, i.e. we want to have a meaningful ability to distinguish between convergent and divergent sequences. Specifically, we want a notion analogous to Cauchy convergence, which we can handle via the satisfaction of

$$\forall \epsilon \exists \alpha ((m, n \geq \alpha) \to (|a_n - a_m| < \epsilon))$$

and

$$\forall \epsilon > 0 \exists \delta > 0 (|x - y| < \delta \to (|f(x) - f(y)| < \epsilon)$$

for sequences of surreal numbers and for surreal-valued functions.

There are two obstacles to this goal:

1. the existence of **gaps** in the surreal numbers;

2. potential discrepancies between analyzing sequences of On length, with the satisfaction of first order sentences in elementary submodels with respect to sentences with functions symbols that are interpreted to be surreal-valued, even though we may interpret ϵ as ranging over all positive *surreal* numbers.

Gaps 'between' surreal numbers have been studied since [5], where Conway considers the games that can be defined with birthday On. Gaps are precisely the games that emerge. This necessitates making a commitment to a meta-theory where Classes are primitive objects: the two most popular being NBG or MK. Without such a commitment, defining gaps is impossible. Unfortunately, even if a commitment is made explicit, conceptual discrepancies creep in by abuse of notation.

While [5], [3], and others make commitments to working in NBG, the game notation is abused when describing Dedekind completions. Specifically within the class of surreal numbers, a game with Left options L and Right options R is identified with the game value $\mathsf{L}|\mathsf{R}$, which we treat as the minimal realization of the cut $(\mathsf{L}|\mathsf{R})$. Specifically, the corresponding cut consists of formula of the form $l < x$ and $x < r$ for $l \in \mathsf{L}$ and $r \in \mathsf{R}$, and the minimal set-theoretic rank in the case of simplicity corresponds to the generator of the convex Class $\mathcal{S}(a)$, which is precisely the class realizing the partial type defined by the cut $(\mathsf{L}|\mathsf{R})$. However, in the current literature we consider cuts that are formed by taking an ordered partition of No into two pieces. A naive approach here then yields cases where we introduce a gap between $\{x : x \leq 0\}$ and $\{y : y > 0\}$, when in the seminal work of Dedekind, the simplest realization of the cut implied by those two Classes would be 0. We introduce a taxonomy of cuts below and construct a Dedekind completion operator and modify the definition of limit presented in [3] to put these ideas in harmony.

Overcoming our second obstacle amounts to extending the transfer principle from Robinson's non-standard analysis as it applies to all hyperreal number systems to the Class of surreal numbers proper. In non-standard analysis, propositions in the language $\mathcal{L}_{\mathsf{or}}$ are true at a set-theoretic level with respect to internal sets; in extending non-standard analysis to No, we must make sure that the interpretation of propositions is true at a Class theoretic level, which in our general setting will be the NBG definable Class extending the underlying set (see Chapter 2.8 for a discussion of interpreting the truth of formula in a Class).

After overcoming our two obstacles, we will re-derive the major results of [3], culminating in a proof that despite being a totally disconnected space, the surreal numbers admit the Intermediate Value Property for functions.

Recalling from Chapter 2.2, an interval I of games in PG is a class such that whenever $\mathsf{x}, \mathsf{y} \in$ I such that $\mathsf{y} \leq \mathsf{x}$ for all $\mathsf{G} \in$ PG (or for any other convex subclass, such as No). Following Lurie's proof, we can bound every partizan game by an ordinal. In fact, we can bound every game with a surreal number. This leads to the following definition:

Definition 32. *For any* $\mathsf{G} \in \mathrm{PG}$*, the* ***Dedekind section*** *of* G *in the sense of Conway [5] is defined as the pair of Classes* $\langle \mathfrak{X}, \mathfrak{Y} \rangle$ *such that* $\mathfrak{X} \cup \mathfrak{Y} = \mathrm{No}$ *and* $\mathfrak{X} > \mathfrak{Y}$*. In particular, this means* $\mathfrak{X}, \mathfrak{Y}$ *are intervals.*

We define the Left and (respectively Right) Dedekind sections as follows:

$$\mathfrak{L}(G) = \langle \{x \in \mathrm{No} \colon x \lhd\mid G\}, \{x \in \mathrm{No} \colon G \leq x\} \rangle$$

$$\mathfrak{R}(G) = \langle \{x \in \mathrm{No} \colon x \leq G\}, \{x \in \mathrm{No} \colon G \lhd\mid x\} \rangle$$

Sections $(\mathfrak{X}, \mathfrak{Y})$ *are* ***numeric*** *if they are of the form* $\mathfrak{L}(G)$ *or* $\mathfrak{R}(G)$ *for* $G \in \mathrm{No}$*.*

Remark 11. *The definition of Dedekind sections in the sense of Conway is helpful for the studying the temperature theory of general Partizan games, as it can be shown that every partizan game* G *is bounded by some ordinal* $\alpha \in \mathrm{On}$*, i.e.* $-\alpha \leq G \leq \alpha$*.*

There is no particular reason why our analysis of the surreal numbers and Partizan games must stop with respect to the construction of games that have ordinal length. In particular, following Conway, we can consider the games created on Day On to be those with at least one option Class. This leads to the following definition:

Definition 33. *Let* $\mathfrak{L}$ *and* $\mathfrak{R}$ *be two subclasses of* No *such that* $\mathfrak{L} \cup \mathfrak{R} = \mathrm{No}$*. Following [5], if* $\mathfrak{L} < \mathfrak{R}$*, then a* ***gap*** *is defined if there does not also exist some* $a \in \mathfrak{L}$ *and* $b \in \mathfrak{R}$ *such that* $a \geq b$*. In these instances, we let* g *denote the gap defined by* $\mathfrak{L}|\mathfrak{R}$*, such that* $\mathfrak{L} < g < \mathfrak{R}$*.*

Gaps can be understood as Dedekind sections on No itself. As mentioned above, they are born on day On, while in terms of tree-rank, they're precisely branches of length On. We will return to the tree-rank notion shortly.

Furthermore, Conway [5] established that gaps have normal forms of the following two types:

Definition 34. *All gaps $\mathfrak{g}$ are of one of the following two types:*[1]

Type I $\sum_{\text{On}} \omega^{y_i} r_i$, *where* (y_i) *is a descending sequence of surreals, and* r_i *are non-zero reals.*

Type II $\sum_{\alpha} \omega^{y_i} r_i \oplus (\pm\omega^{\Theta})$, *with* (y_i) *and* (r_i) *as above, and* Θ *a gap whose Right option Class contains all of the* y_i, *and* $\oplus$ *denotes the sum of a number* a *and a gap* g, *i.e.* $a \oplus g = \{a + g^{\mathfrak{L}}\} \mid \{a + g^{\mathfrak{R}}\}$, *and* $\omega^{\Theta} = \{0, a\omega^{l}\} \mid \{b\omega^{r}\}$, *and* $a, b \in \mathbb{R}_{>0}$, $l \in \mathfrak{L}_{\Theta}$, *and* $r \in \mathfrak{R}_{\Theta}$.

Remark 12. *The following are named gaps which are helpful in guiding the notions of limit and completion for the purpose of analysing surreal-valued functions:*

1. $\text{On} = \text{No}|\emptyset = \omega^{\text{On}}$. *The Class of ordinals is the gap larger than all the surreal numbers-it is properly speaking, the surreal notion of infinity.*
2. $\frac{1}{\text{On}} = \text{No}_{\leq 0}|\text{No}_{>0} = \omega^{-\text{On}}$ *is the gap between 0 and all positive numbers.*
3. $\text{Off} = \emptyset|\text{No} = -\text{On}$. *The gap smaller than all surreal numbers is the surreal notion of negative infinity.*
4. $\infty = (\text{No}_{<0} \cup \text{No}^{\leq 0})|\text{No}^{\geq 0}_{>0} = \omega^{1/\text{On}}$. *This object is the infinity used in real analysis.*
5. $\frac{1}{\infty} = \text{No}^{<0}|\text{No}^{\geq 0}_{>0} = \omega^{-1/\text{On}}$, *the gap between the infinitesimals and the positive reals.*

Just as we may take the Dedekind completion of any linearly ordered set, we can take the Dedekind completion of No:

Definition 35. *The* ***Dedekind completion*** *of* No, *denoted by* $\text{No}^{\mathfrak{D}}$, *contains all numbers and (analytic) gaps, which we identify by their corresponding Dedekind representations.*

---

[1] *Not a type in the sense of Model theory nor in the sense of Type theory.*

Remark 13. *The surreal numbers differ from* $\mathbb{R}$ *in the sense that the surreals are demonstrably not Dedekind complete, whereas the real numbers are. However, the basic arithmetic operations on* $\mathrm{No}^{\mathfrak{D}}$ *other than negation must be defined differently from* No*.*

We will return to how exactly we may define $\mathrm{No}^{\mathfrak{D}}$ and what we mean by analytic gaps after we discuss cuts in greater detail.

Following the exposition on Cuesta-Dutari cuts found in [16], we have the following definitions:

Definition 36. *For any ordered set* X*, if* $\mathsf{L}, \mathsf{R} \subset \mathsf{X}$ *such that* $\mathsf{L} < \mathsf{R}$*, then the pair* (L, R) *is a* ***Conway cut in*** X*. Whenever* (L, R) *is a Conway cut such that* $\mathsf{L} \cup \mathsf{R} = \mathsf{X}$*, then* (L, R) *is a* ***Cuesta Dutari cut****. Finally, if* (L, R) *is a Cuesta-Dutari cut such that* L, R *are both non-empty Classes in* NBG*, then* (L, R) *is a* ***Dedekind cut*** *or cut for short. For any Dedekind cut* (L, R) *in* X*, if* L *has a greatest element* a *or* R *has a least element* b*, then* (L, R) *is* ***rational****, and* a *or* b *is a* ***cut point*** *of* (L, R)*. Otherwise,* (L, R) *is a* ***gap****.*

Notation 3. *We reserve* $(\mathsf{L}|\mathsf{R})$ *to denote Conway cuts in* No*. Specifically, we use* $(\mathsf{L}|\mathsf{R})$ *to denote the partial type consisting of formula* $\mathsf{l} < \mathsf{x} < \mathsf{r}$ *as* $\mathsf{l} \in \mathsf{L}$ *and* $\mathsf{r} \in \mathsf{R}$*, and* $\mathsf{L}|\mathsf{R}$ *to denote the simplest realization of the cut, namely, the game value/ surreal number such that* (L, R) *is cofinal with the position closed canonical form of the minimal realization of* $(\mathsf{L}|\mathsf{R})$*. We will extend this notation to gaps as well.*

Cuesta-Dutari cuts are worth developing in some detail as they provide an alternate (and historically earlier) development of the surreal numbers.

2.4.0.1 Filling in Conway Cuts

It is instructive to recall Conway's simplicity theorem [5].

Theorem 9. *Let* $L, R \subset \mathrm{No}$ *such that* $L < R$. *Let* $I = \{y \in \mathrm{No} \colon L < y < R\}$. *Then* I *is a non-empty convex class for which there exists a unique* $x \in I$ *such that* $\iota(x) < \iota(y)$ *for all* $y \in I \backslash \{x\}$.

Proof. See Theorem 11 of [5] for details. □

We have summarized this result earlier in our introduction, but are including it as a separate result here as a reminder.

Definition 37. *Let* $L, R \subset \mathrm{No}$ *such that* $L < R$. *We denote the Conway cuts of* No *by* $(L|R)$. *Let* x *be the simplest number satisfying* $L < x < R$. *We say* $(L|R)$ *is **timely** if* $L, R \subseteq O_{\iota x}$*; the cuts associated with the position closed canonical form of* x *are always timely.*

Let $\mathfrak{E} = \{(L|R) \colon L, R \subset_{\text{subset}} \mathrm{No} \wedge L < R\}$ *be the Class of proper Conway cuts. Let* $\mathfrak{E}^* = \{(L|R) \colon L, R \subset \mathrm{No} \wedge L < R\}$ *denote the Class of Cuts of* No. *For every* $(L|R) \in \mathfrak{E}^*$, *let* $\mathcal{S}(F|G) = \{y \in \mathrm{No} \colon F < y < G\}$.

Proposition 5. *For all* $a \in \mathrm{No}$, $\mathcal{S}(a) = \mathcal{S}(L_a|R_a)$ *and for all* $(F|G) \in \mathfrak{E}$ *such that* $a = F|G$, *we have* $\mathcal{S}(F|G) \subset \mathcal{S}(a)$.

Proof. These results follow from Conway's Simplicity theorem and the convexity of $\mathcal{S}(a)$ and $\mathcal{S}(F|G)$. In particular,

$$\mathcal{S}(a) = \{y \in \mathrm{No} \mid a \leq_s y\} = \{y \in \mathrm{No} \mid L_a < y < R_a\} = \mathcal{S}(L_a|R_a),$$

with equality following from $\mathfrak{a}$ being the simplest element realizing $(\mathsf{L}_\mathfrak{a}|\mathsf{R}_\mathfrak{a})$. Furthermore, since (F,G) is cofinal in $(\mathsf{L}_\mathfrak{a},\mathsf{R}_\mathfrak{a})$, we have for every $\mathsf{f}\in\mathsf{F}$ (and similarly for $\mathfrak{g}\in\mathsf{G}$) such that for every $\mathfrak{a}^\mathsf{L}\in\mathsf{L}_\mathfrak{a}$ (or $\mathfrak{a}^\mathsf{R}\in\mathsf{R}_\mathfrak{a}$ in case of G), $\mathfrak{a}^\mathsf{L}<\mathsf{f}$, then $\mathsf{f}\in\mathcal{S}(\mathfrak{a})$, but $\mathsf{f}\notin\mathcal{S}(\mathsf{F}|\mathsf{G})$ by construction. On the other hand, if $\mathfrak{y}\in\mathcal{S}(\mathsf{F}|\mathsf{G})$, then $\mathsf{F}<\mathfrak{y}<\mathsf{G}$ by construction, and then by Conway's simplicity theorem, $\mathfrak{a}\leq_s\mathfrak{y}$. □

Theorem 10. *For every* $(\mathsf{L}|\mathsf{R})\in\mathfrak{E}^*$*,* $|\mathcal{S}(\mathsf{L}|\mathsf{R})|$ *is exactly* $0,1$ *or* On*.*

Proof. If $\mathsf{L}\cup\mathsf{R}=\mathrm{No}$, then $\mathcal{S}(\mathsf{L}|\mathsf{R})$ will be empty by definition, as there will be no surreal number between L and R. Now suppose that $\mathsf{L}\cup\mathsf{R}\neq\mathrm{No}$. By Conway's simplicity theorem and the convexity of $\mathcal{S}(\mathsf{L}|\mathsf{R})$, we know that there is a simplest element $\mathfrak{a}\in\mathcal{L}\mathsf{R}$ (in fact $\mathfrak{a}=\mathsf{L}|\mathsf{R}$). So $|\mathcal{S}(\mathsf{L}|\mathsf{R})|\geq 1$. However, if $|\mathcal{S}(\mathsf{L}|\mathsf{R})|>1$, then by convexity, there are Class many surreal number satisfying $\mathsf{L}<x<\mathsf{R}$. If not, then, without loss of generality, suppose $\mathsf{L}\neq\{x\colon x<\mathfrak{a}\}$, and let $\mathfrak{g}$ denote the supremum of L ($\mathfrak{g}$ may be a gap, or it may be a surreal number). If $\mathfrak{g}$ is a surreal number less than $\mathfrak{a}$, then by convexity the Class of sign sequences $\langle(\mathfrak{g})\frown\beta(i)\colon i\in\mathrm{On}^+\rangle$ will be a subClass $\mathcal{S}(\mathsf{L}|\mathsf{R})$. Similarly, if $\mathfrak{g}$ is a gap, depending on the type of gap, we can find some surreal number with Conway normal form that is infinitesimally larger than the Conway normal form for $\mathfrak{g}$ but less than the Conway normal form of $\mathfrak{a}$, and then apply the convexity argument again. □

2.4.0.2 Dedekind representations and the Dedekind Completion

Definition 38. *For every* $\mathfrak{a}\in\mathrm{No}$*, the **Dedekind representation** of* $\mathfrak{a}$ *is the cut* $(\mathfrak{L}_\mathfrak{a}|\mathfrak{R}_\mathfrak{a})$*, where* $\mathfrak{L}_\mathfrak{a}=\{\mathfrak{y}\in\mathrm{No}\colon\mathfrak{y}<\mathfrak{a}\}$ *and* $\mathfrak{R}_\mathfrak{a}=\{\mathfrak{y}\in\mathrm{No}\colon\mathfrak{a}<\mathfrak{y}\}$*.*

Each Dedekind representation is definable as a Class in NBG. One of the chief insights in Rubinstein-Salzedo and Swaminathan [3] was to push the analysis of surreal numbers up to the statements about the Dedekind representations of numbers and gaps. We attempt to reconcile their insights within simplicity framework whereby we consistently select for the minimal rank realization of cuts. However, this will also require that all rational Cuesta-Dutari cuts of No correspond to gaps (importantly not the rational Cuesta-Dutari cuts defining particular surreal numbers).

It is an important result that No is not Dedekind complete. In particular, we can see this when examining inf and sup over the Class of No with respect to unbounded *sets*.

For example, consider the set $A = \{-\frac{1}{n+1} : n \in \omega\}$. Intuitively in $\mathbb{R}$, we would say $\sup A = 0$. However, $-\frac{1}{\omega}$ is strictly less than 0 and strictly greater than $-\frac{1}{n}$. In fact, there's an entire class of values that we can say is rooted at 0 that would be above $-\frac{1}{n+1}$ for all $n \in \omega$ and below 0. So there is no least upper bound for A in No. That is, $\sup A = \mathfrak{g}$ for a gap $\mathfrak{g}$.

To assist in our analysis, we will introduce the following values

Definition 39. *For all* $A \subseteq \text{No}$*, we define the classes*

$$\mathfrak{L}_{r.\sup A} = \{y \in \text{No}\colon \exists a \in A(y \leq a)\}$$

$$\mathfrak{R}_{r.\inf A} = \{y \in \text{No}\colon \exists a \in A(a \leq y)$$

We now state and prove several novel, if elementary theorems, that will inform our re-derivation of the results of [3] in Chapter 2.4.0.3.

Theorem 11. *For all* $a \in \mathrm{No}$*, and every* $(F|G) \in \mathfrak{E}^*$ *such that* $F|G = a$*,* $(\mathfrak{L}_a, \mathfrak{R}_a)$ *is cofinal in* (F, G)*.*

Proof. This immediately follows from $F \subset \mathfrak{L}_a$ and $G \subset \mathfrak{R}_a$. □

Theorem 12. *Suppose* $A, B \subset \mathrm{No}$ *such that* $A \subset B$*. Then*

$$\mathfrak{L}_{r.\sup A} \subset \mathfrak{L}_{r.\sup B}$$

and

$$\mathfrak{R}_{r.\inf B} \subset \mathfrak{R}_{r.\inf A}$$

Proof. Suppose $A \subsetneq B$ and let $b \in B \backslash A$ such that $b > a$ for all $a \in A$. If $x \in \mathrm{No}$ such that $x \leq b$, then $b \in \mathfrak{L}_{r.\sup B} \backslash \mathfrak{L}_{r.\sup A}$. On the other hand, if $x \leq a$ for some $a \in A$, then $x \in \mathfrak{L}_{r.\sup B}$, whence we have the desired inclusion. Similarly for the Right Classes. □

Theorem 13. *Let* $\langle (F_\alpha | G_\alpha) \rangle_{\alpha \in \mathrm{On}}$ *be an* On *length sequence of Conway cuts in* $\mathfrak{E}$ *such that for all* $\alpha, \beta \in \mathrm{On}$

1. $F_\alpha < G_\beta$*;*

2. $F_\alpha \subset F_\beta$ *and* $G_\alpha \subset G_\beta$ *for all* $\beta \ni \alpha$*.*

Then

1. $(\bigcup_{\alpha \in \mathrm{On}} F_\alpha | \bigcup_{\alpha \in \mathrm{On}} G_\alpha)$ *is realized by* $\bigcap_{\alpha \in \mathrm{On}} \mathcal{S}(F_\alpha | G_\alpha)$*;*

2. $\bigcap_{\mathrm{On}} \mathcal{S}(F_\alpha | G_\alpha)$ *is empty if and only if* $\bigcup F_\alpha | \bigcup G_\alpha$ *is a gap.*

Proof. 1. $x \in \bigcap \mathcal{S}(F_\alpha|G_\alpha)$ if and only if $F_\alpha < x < G_\alpha$ for all $\alpha \in \textsc{On}$ if and only if $\bigcup F_\alpha < x < \bigcup G_\alpha$.

2. Let $x_\alpha = F_\alpha|G_\alpha$. Since $F_\alpha \subset F_\beta$ (and similarly for G_α) for all $\alpha \in \beta$, it is immediate that for all $\beta \ni \alpha$, $F_\alpha < x_\beta < G_\beta$. So we have $\mathcal{S}(x_\alpha) \supset \mathcal{S}(x_\beta)$. Moreover, we have $\mathcal{S}(F_\alpha|G_\alpha) \supset \mathcal{S}(F_\beta|G_\beta)$ for $\beta \ni \alpha$ since

$$\mathfrak{L}_{\text{r.}\sup F_\alpha} \subseteq \mathfrak{L}_{\text{r.}\sup F_\beta} < \mathfrak{R}_{\text{r.}\inf G_\beta} \subseteq \mathfrak{R}_{\text{r.}\inf G_\alpha}$$

by Theorem 12, and $x_\beta \in \mathcal{S}(F_\alpha|G_\alpha)$.

Supposing now that $\bigcap_{\textsc{On}} \mathcal{S}(F_\alpha|G_\alpha) = \emptyset$. Then for every x_α there is a $\beta \ni \alpha$ such that $x_\alpha \leq f_\beta$ or $g_\beta \leq x_\alpha$ for either some $f_\beta \in F_\beta$ or respectively some $g_\beta \in G_\beta$. We can extend $\bigcup F_\alpha$ to $\mathfrak{L}_{\text{r.}\sup \bigcup F_\alpha} = \mathfrak{L}$ and $\bigcup G_\alpha$ to $\mathfrak{R}_{\text{r.}\inf \bigcup G_\alpha} = \mathfrak{R}$ such that

$$(\mathfrak{L}|\mathfrak{R}) = (\bigcup F_\alpha | \bigcup G_\alpha).$$

It follows by part (1) that $\mathfrak{L} \cup \mathfrak{R} = \textsc{No}$ (as the cut cannot be realized by any surreal number). But then by definition, $\mathfrak{L}|\mathfrak{R}$ defines a gap.

The reverse direction is immediate since $\mathcal{S}(F|G) \subset \textsc{No}$.

□

Definition 40. *Let $\mathfrak{X}$ be a space of Conway cuts. Let $[\mathfrak{X}]$ denote the space of realizations of cuts with minimal set-theoretic rank up to* $\textsc{On}$.

In particular $[\mathfrak{E}^*\backslash\mathfrak{E}]$ *can be identified with branches in the tree* $2^{\leq \mathrm{On}}$*, where* $\mathfrak{a} \in 2^{\leq \mathrm{On}}$ *satisfies* $\mathfrak{F} < \mathfrak{a} < \mathfrak{G}$*, whenever* $\mathfrak{F}$ *or* $\mathfrak{G}$ *is a proper Class.*

We define the ***Cuesta-Dutari operator*** $(-)^{\mathfrak{N}} : \mathfrak{E}^* \to [\mathfrak{E}^*\backslash\mathfrak{E}]$ *by*

$$(\mathsf{F}|\mathsf{G}) \mapsto \mathfrak{L}_{\mathrm{r.\,sup}\,\mathsf{F}}|\mathfrak{R}_{\mathrm{r.\,inf}\,\mathsf{G}}.$$

We define the ***Dedekind operator*** $(-)^{\mathfrak{D}} : \mathfrak{E}^* \to [\mathfrak{E}^*\backslash\mathfrak{E}]$ *by*

$$(\mathsf{F}|\mathsf{G}) \mapsto \mathfrak{L}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}|\mathfrak{R}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}$$

In particular, the image of $(-)^{\mathfrak{D}}$ *identifies the proper Class cuts with Left options coinitial with* No *and Right options cofinal with* No*.*

Definition 41. *An* ***analytic gap*** $\mathfrak{g}$ *is represented by any* $(\mathsf{F}|\mathsf{G}) \in \mathfrak{E}^*$ *such that there does not exist an* $\mathfrak{a} \in \mathrm{No}$ *so that* $(\mathsf{F}|\mathsf{G})^{\mathfrak{D}} = \mathfrak{a}$*.*

Theorem 14. $\mathfrak{N}$ *and* $\mathfrak{D}$ *are well-defined operators.*

Proof. If $(\mathsf{F}|\mathsf{G}) = (\mathsf{L}|\mathsf{R})$, then for all $\mathsf{x} \in \mathrm{No}$

$$\mathsf{F} < \mathsf{x} < \mathsf{G} \iff \mathsf{L} < \mathsf{x} < \mathsf{R}.$$

But then

$$\mathfrak{L}_{\mathrm{r.\,sup}\,\mathsf{F}} = \mathfrak{L}_{\mathrm{r.\,sup}\,\mathsf{L}}$$

and similarly for G and R. But since we've not changed the underlying space of surreal numbers satisfying the cut, we find that

$$(\mathsf{F}|\mathsf{G})^{\mathfrak{N}} = (\mathsf{L}|\mathsf{R})^{\mathfrak{N}},$$

whence $(\mathsf{F}|\mathsf{G})^{\mathfrak{D}} = (\mathsf{L}|\mathsf{R})^{\mathfrak{D}}$. □

Theorem 15. *For all* $(\mathsf{F}|\mathsf{G}) \in \mathfrak{E}^*$, $(\mathsf{F}|\mathsf{G})$ *represents a gap if and only if* $\mathfrak{L}_{\mathsf{r}.\sup \mathsf{F}} = \mathfrak{L}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}$ *and* $\mathfrak{R}_{\mathsf{r}.\inf \mathsf{G}} = \mathfrak{R}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}$

Proof. Let $(\mathsf{F}|\mathsf{G}) \in \mathfrak{E}^*$. It is immediate by construction that

$$\mathfrak{L}_{\mathsf{r}.\sup \mathsf{F}} \subset \mathfrak{L}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}$$

and

$$\mathfrak{R}_{\mathsf{r}.\inf \mathsf{F}} \subset \mathfrak{R}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}.$$

Suppose now that at least one of these is a strict containment. Without loss of generality, suppose

$$\mathfrak{L}_{\mathsf{r}.\sup \mathsf{F}} \subsetneq \mathfrak{L}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}}.$$

From here, we find that there are Class many $\mathsf{a} \in \text{No}$ such that

$$\mathfrak{L}_{\mathsf{r}.\sup \mathsf{F}} < \mathsf{a} < \mathfrak{R}_{\mathsf{r}.\inf \mathsf{G}}$$

But by Theorem 13, it follows that $(\mathsf{F}|\mathsf{G})$ defines a number. Importantly, we may have equality up to exactly one of the two Options, but we need at least one of the two options to be a strict containment.

If $(\mathsf{F}|\mathsf{G})$ represents a gap, then $\mathfrak{L}_{r.\sup \mathsf{F}} \cup \mathfrak{R}_{r.\inf \mathsf{G}} = \mathrm{No}$, as neither of the two options are a strict containment, from which the forward direction immediately follows. □

Theorem 16. *For all* $(\mathsf{F}|\mathsf{G}) \in \mathfrak{E}^*$, $(\mathsf{F}|\mathsf{G})^{\mathfrak{N}} = (\mathsf{F}|\mathsf{G})^{\mathfrak{D}}$.

Proof. If $(\mathsf{F}|\mathsf{G})$ represents a surreal number a, by Theorem 15, this is immediate, since $\mathfrak{L}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}} \cup \mathfrak{R}_{(\mathsf{F}|\mathsf{G})^{\mathfrak{N}}} \subsetneq \mathrm{No}$. If $(\mathsf{F}|\mathsf{G})$ represents a gap, this follows immediately by Theorem 15. □

Remark 14. *As we may readily identify* $\mathfrak{E}^*$ *as the class of all cuts of* No*, and since every cut representing a number* a *is taken to the corresponding Dedekind representation of* a*, we may denote by* $\mathrm{No}^{\mathfrak{D}}$ *the Dedekind completion of the surreal numbers as desired.*

Precisely, the Dedekind completion of the surreal numbers will be the Class containing all surreal numbers and all analytic gaps. Although the Dedekind completion is not a Class in NBG, we can use it as an abbreviation for a well-formed formula in NBG with a free Class variable.

In [3], the authors show that On-length sequences are necessary in order to obtain a standard $\delta\epsilon$ notion of convergence. Namely, by Theorem 16 in [3], for any $a \in \mathrm{No}$, there do not exist eventually non-constant sequences (t_n) of limit-ordinal length α such that for every surreal $\varepsilon > 0$, there is a $\beta \in \alpha$ such for all $\gamma \geq \beta$, $|t_\gamma - a| < \varepsilon$. See [3, 8] for a full discussion on the pathologies that arise when considering non-ordinal length sequences, but following Sikorski, we need sequences of length On to obtain convergence for nontrivial sequences of surreal numbers, as convergent nontrivial sequences must at least be of length ω_μ in fields with character ω_μ, where ω_μ is an initial regular ordinal (and No has character On).

Definition 42. *Let $\mathfrak{A}$ be a sequence of surreal numbers of* On *length. Let*

$$X = \left\{ p \in \mathrm{No} \colon p < \sup \left(\bigcup_{i \in \mathrm{On}} \bigcap_{j \geq i} \mathfrak{L}_{a_j} \right) \right\}$$

$$Y = \left\{ q \in \mathrm{No} \colon q > \inf \left(\bigcup_{i \in \mathrm{On}} \bigcap_{j \geq i} \mathfrak{R}_{a_j} \right) \right\}$$

We define the limit of $\mathfrak{A}$

$$\mathcal{L}(\mathfrak{A}) = (X|Y)^{\mathfrak{D}}$$

$\mathfrak{A}$ *is a* ***Cauchy sequence*** *if for every surreal $\epsilon > 0$, there exists $\alpha \in \mathrm{On}$ such that for all $i, j \in \mathrm{On} \setminus \alpha$, $|a_i - a_j| < \epsilon$.*

One immediate consequence of the above definition is that $\mathrm{No}^{\mathfrak{D}}$ will not contain certain kinds of Type II gaps. In particular, the definition given above omits gaps like $\frac{1}{\mathrm{On}}$. An immediate motivation for this is that we wish to preserve the intuition that a sequence like $(\frac{1}{\alpha+1})_{\alpha \in \mathrm{On}}$ ought to converge to 0 under the classical ϵ-δ definition of a limit.

2.4.0.3 Fundamentals of Surreal Analysis

One of the key results of [3] is the proof that despite No being a totally disconnected space, surreal-valued genetic functions nonetheless satisfy the intermediate value property. We aim to summarize the definitions and results necessary to arrive at this conclusion.

While we will follow Rubinstein-Salzedo and Swaminathan's proposed "topology" (see [3]), there are two immediate issues that need to be addressed prior to any further elaboration. The first is that this is not a conventional topology in that we will not be considering our basis to consist of sets, but of proper Classes, and secondly, rather than take arbitrary unions, we are taking arbitrary proper set sized unions. We will discuss an issue raised about doing the latter, once we define the review the topology $\mathcal{T}$ on $\textsc{No}$ proposed in [3].

Let $\mathcal{T}$ be a collection of subclasses of $\textsc{No}$ such that:

1. $\emptyset, \textsc{No} \in \mathcal{T}$;

2. $\bigcup\limits_{I} A_i \in \mathcal{T}$ for any subcollection $\{A_i\}_{i\in I} \subset \mathcal{T}$ over a proper index set I;

3. $\bigcap\limits_{[n]} A_i \in \mathcal{T}$ for any finite subcollection $\{A_i\}_{i\in[n]} \subset \mathcal{T}$ over a finite set $[n]$.

All such $\mathcal{T}$ are collections of open Classes. The standard topology is generated by the empty set, $\textsc{No}$, and all non-empty subintervals. Precisely, $\mathcal{I}$ is a non-empty subinterval if:

1. $\mathcal{I}$ has endpoints in $\textsc{No} \cup \{\textsc{On}, \textsc{Off}\}$;

2. $\mathcal{I}$ does not contain its endpoints.

In particular, a subclass $A \subset \textsc{No}$ is open whenever if has the form $A = \bigcup\limits_{j\in J} \mathcal{I}_j$, where J is a proper set, and $\mathcal{I}_j$ is an open interval.

For any $A \subset \textsc{No}$, a function $f : A \to \textsc{No}$ is continuous on A with respect to $\mathcal{T}$ if for any Class $B \in \mathcal{T}$, $f^{-1}(B)$ is open in A.

Remark 15. *One reviewer of this article has mentioned that $\mathcal{T}$ above is not properly a topology given that we are not allowing unrestricted unions. The complaint arises if we are to replace*

$\textsc{No}$ with $2^{<\kappa}$ where κ is some inaccessible cardinal. This warrants further investigation, since Rubinstein-Salzedo and Swaminathan explicitly discuss the restriction to sets as being necessary in order to ensure connectedness and compactness arguments (see the remark after Definition 6 of [3]). In particular, it warrants investigation into what amendments may be necessary to the above definition so that for restrictions to $2^{<\kappa}$ for an inaccesible κ, we can replicate the proofs for connectedness and compactness from [3].

Until said work is done, we will advise the reader that should this restriction to unions of proper sets prove bothersome when allowing for families of proper Classes, take comfort that we do not actually use the later results from [3] concerning this topology, and have only included it for the sake of being comprehensive in our amendments to [3].

Remark 16. *While $\mathfrak{L}(f)$ can be defined for surreal valued functions without genetic definitions, we are only concerned about functions with genetic definitions.*

[3] proved the following characterization of all \$On\$-length sequences in $\textsc{No}^{\mathfrak{D}}$.

Lemma 2. *If $\mathfrak{A}$ is an $\textsc{On}$-length sequence in $\textsc{No}$, then either:*

1. *If $\mathfrak{A}$ converges to a Type II gap $\mathfrak{g}$, then $\mathfrak{A}$ is not Cauchy;*
2. *If $\mathfrak{A}$ is Cauchy, and converges to a Type I gap $\mathfrak{g}$, then $\mathfrak{g} = \sum_{\textsc{On}} \omega^{y_i} r_i$ such that $\lim_{i \in \textsc{On}} y_i = \textsc{Off}$.*
3. *Otherwise, if $\mathfrak{A}$ is Cauchy then $\mathfrak{L}(\mathfrak{A}) \in \textsc{No}$.*

Given the redefinition of Dedekind sections relative to the one given in [3], the following is an adjusted definition of the limit of a surreal-valued function at a point $a \in \textsc{No}$.

Definition 43. *Let* $a, b, c \in \mathrm{No}$ *such that* $b < a < c$*, and let* $\mathcal{I} = (b,a) \cup (a,c)$*. For a function* $f : \mathcal{I} \to \mathrm{No}$*, we define the limit of* $f(x)$ *as* $x \to a$ *with the following Conway cuts:*

$$\mathcal{L}(f)[a] := \left(\left\{ p \colon p < \sup \left(\bigcup_{x \in (b,a)} \bigcap_{y \in [x,a)} \mathfrak{L}_{f(y)} \cap \bigcup_{x \in (a,c)} \bigcap_{y \in (a,x]} \mathfrak{L}_{f(y)} \right) \right\} \;\middle|\; \left\{ q \colon q > \inf \left(\bigcup_{x \in (b,a)} \bigcap_{y \in [x,a)} \mathfrak{R}_{f(y)} \cap \bigcup_{x \in (a,c)} \bigcap_{y \in (a,x]} \mathfrak{R}_{f(y)} \right) \right\} \right)$$

Finally, we can describe the limits from the left and right of a *as follows:*

$$\mathcal{L}(f)[a^-] := \left(\left\{ p \colon p < \sup \left(\bigcup_{x \in (b,a)} \bigcap_{y \in [x,a)} \mathfrak{L}_{f(y)} \right) \right\} \;\middle|\; \left\{ q \colon q > \inf \left(\bigcup_{x \in (b,a)} \bigcap_{y \in [x,a)} \mathfrak{R}_{f(y)} \right) \right\} \right)$$

and

$$\mathcal{L}(f)[a^+] := \left(\left\{ p \colon p < \sup \left(\bigcup_{x \in (a,c)} \bigcap_{y \in (a,x]} \mathfrak{L}_{f(y)} \right) \right\} \;\middle|\; \left\{ q \colon q > \inf \left(\bigcup_{x \in (a,c)} \bigcap_{y \in (a,x]} \mathfrak{R}_{f(y)} \right) \right\} \right).$$

We say that the limit, along with the Left and Right hand side limits, are defined whenever they are invariant under the Dedekind representation operator. Otherwise the limit will be undefined.

Remark 17. *This redefinition of the limit of surreal valued functions differs from the definition given in [3] by including the Left options of the function evaluated on intervals whose left endpoint is* a *and including in the Right options the Right options of the functions evaluated on the interval whose right endpoint is* a*. This redefinition is intended to make explicit use of simplicity, the redefinition of Dedekind completions in light of simplicity, and also to evocatively suggest that limits that diverge or the cases where limits do not exist are precisely the cases where* *the function becomes* fuzzy *in the limit. For example, consider* $\frac{1}{x}$ *as* $x \to 0$*. By definition, we would have* $\mathfrak{L}(\frac{1}{x})[0] = (\text{No}|\text{No})^{\mathfrak{D}}$*, which does not exist as* $(\text{No}, \text{No}) \notin \mathfrak{E}$*, but more to the point,* No|No *is sort of the ultimate fuzzy game.*

In contrast, the definition of limit from [3] without overloading the game-cut notation for Dedekind sections would have that the Left options be the Class

$$\left\{ p \colon p < \sup \left(\bigcup_{(\text{Off},0)} \bigcap_{[x,0)} \mathfrak{L}_{\frac{1}{y}} \right) \right\}$$

and the Right options be the Class

$$\left\{ q \colon q > \inf \left(\bigcup_{(0,\text{On})} \bigcap_{(0,x])} \mathfrak{R}_{\frac{1}{y}} \right) \right\}$$

But both Classes are empty when drawing p, q *from* No*, and so the interpretation of these games under the definition of* $(-)^{\mathfrak{D}}$ *would be that the limit of* $\frac{1}{x}$ *as* $x \to 0$ *is 0, which is absurd. We thus motivate our proposed redefinition in order to better reason about the limits of surreal valued functions in light of our other definitions.*

Finally, it should be noted that the definition of Dedekind completion given in [3] would suggest that because the Left and Right options are empty, the limit doesn't exist. So the notion of limit in that paper is not incorrect, only that it is incompatible with the notion of the limit being defined with respect to simplicity*.*

Lemma 3. *For all* $a, b, c, d, e \in \text{No}$ *such that* $a < b < c < d < e$ *and any surreal valued function defined on* (a, e) *or* $(a, c) \cup (c, e)$*, let* $g := f \upharpoonright (b, c) \cup (c, d)$*. Then* $\mathfrak{L}(f)[c] = \mathfrak{L}(g)[c]$*.*

Proof. This proof is nearly identical to the proof of Lemma 21 from [3]. The main differences are instead of considering sequences of ON-length, we're evaluating the monotonicity properties of the sup and inf operations on image of Class-sized intervals under the function f.

To prove this we need to show that

1. $\sup\left(\bigcup_{x\in(a,c)}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}\right) = \sup\left(\bigcup_{x\in(b,c)}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}\right)$
2. $\sup\left(\bigcup_{x\in(c,e)}\bigcap_{y\in(c,x]}\mathfrak{L}_{f(y)}\right) = \inf\left(\bigcup_{x\in(c,d)}\bigcap_{y\in(c,x]}\mathfrak{L}_{f(y)}\right)$
3. $\inf\left(\bigcup_{x\in(a,c)}\bigcap_{y\in[x,c)}\mathfrak{R}_{f(y)}\right) = \sup\left(\bigcup_{x\in(b,c)}\bigcap_{y\in[x,c)}\mathfrak{R}_{f(y)}\right)$
4. $\inf\left(\bigcup_{x\in(c,e)}\bigcap_{y\in(c,x]}\mathfrak{R}_{f(y)}\right) = \inf\left(\bigcup_{x\in(c,d)}\bigcap_{y\in(c,x]}\mathfrak{R}_{f(y)}\right)$

It will suffice to prove (1), since the proofs for (2)-(4) are nearly identical. Let $M = \left(\bigcup_{x\in(a,c)}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}\right)$ and $N = \left(\bigcup_{x\in(b,c)}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}\right)$. Since for all $a < x \leq z < c$, we have $\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)} \subseteq \bigcap_{y\in[z,c)}\mathfrak{L}_{f(y)}$, it follows that $P = \left(\bigcup_{x\in(a,b]}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}\right) \subseteq N$. However, $P \cup N = M$, and so $N = M$. Finally, it follows that $\sup M = \sup N$.

Next, by (1)-(4), it will suffice to set

$$M_1 = \bigcup_{x\in(a,c)}\bigcap_{y\in[x,c)}\mathfrak{L}_{f(y)}$$

$$M_2 = \bigcup_{x\in(c,e)}\bigcap_{y\in(c,x]}\mathfrak{L}_{f(y)}$$

$$S_1 = \bigcup_{x\in(a,c)}\bigcap_{y\in[x,c)}\mathfrak{R}_{f(y)}$$

$$S_2 = \bigcup_{x\in(c,e)} \bigcap_{y\in(c,x]} \mathfrak{R}_{f(y)}$$

and let $M = M_1 \cap M_2$ and $S = S_1 \cap S_2$. Further, it is immediate that $\sup M = \min\{\sup M_1, \sup M_2\}$ and $\inf S = \max\{\inf S_1, \inf S_2\}$, from which we find

$$\mathcal{L}(f)[c] = (\{p\colon p < \sup M\} | \{q\colon q > \inf T\})^{\mathfrak{D}} = \mathcal{L}(g)[c].$$

□

Lemma 4. *For* $a, b, c \in \mathrm{No}$ *such that* $b < a < c$*, and* $f : (b, a) \cup (a, c) \to \mathrm{No}$*,*

1. $\mathcal{L}(f)[a^-] = \mathfrak{l} \in \mathrm{No}$ *if and only if for*

$$M_1 = \bigcup_{x\in(a,c)} \bigcap_{y\in[x,c)} \mathfrak{L}_{f(y)}$$

and

$$S_1 = \bigcup_{x\in(a,c)} \bigcap_{y\in[x,c)} \mathfrak{R}_{f(y)},$$

we have $\sup M_1 = \inf S_1 \in \mathrm{No}$*;*

2. *Similarly,* $\mathcal{L}(f)[a^+] \in \mathrm{No}$*, if and only if for*

$$M_2 = \bigcup_{x\in(c,e)} \bigcap_{y\in(c,x]} \mathfrak{L}_{f(y)}$$

and

$$S_2 = \bigcup_{x\in(c,e)} \bigcap_{y\in(c,x]} \mathfrak{R}_{f(y)},$$

we have $\sup M_2 = \inf S_2 \in \mathrm{No}$.

3. $\mathcal{L}(f)[a] \in \mathrm{No}$ *if and only if* $\mathcal{L}(f)[a^-] = \mathcal{L}(f)[a^+] \in \mathrm{No}$.

Proof. It will suffice to prove (1), since the proof for (2) is identical. First, since f is surreal valued on $(b, a) \cup (a, c)$, it follows for all $f(y) \in (b, a) \cup (a, c)$ that we have

$$\mathfrak{L}_{f(y)} \cup \mathfrak{R}_{f(y)} = \mathrm{No}\backslash\{f(y)\}.$$

Now, in the converse direction, if $\sup M_1 = \inf S_1 = \mathfrak{l} \in \mathrm{No}$, then by straightforward application of the definitions, we have $\mathcal{L}(f)[a^-] = (\mathfrak{L}_\mathfrak{l}|\mathfrak{R}_\mathfrak{l})^{\mathfrak{D}} = \mathfrak{l}$.

So now, for the forward direction, we'll prove the contrapositive statement. If we first suppose that $\sup M_1 \neq \inf S_1$, then by the linear ordering of No, it follows either that $\sup M_1 > \inf S_1$ or $\sup M_1 < \inf S_1$. If the former, then we have an immediate contradiction, since then the function would be fuzzy on some subinterval $(x, a) \subset (b, a)$, contradicting that f is surreal-valued on (b, a). Specifically, the function must be fuzzy on some subinterval as some of the Right options of the function would be less than some of the Left options of the function.

So we now suppose that $\sup M_1 < \inf S_1$. Then it follows immediately that we do not have a Cuesta-Dutari cut, and so the limit will be undefined.

Finally, for (3), we note that by definition, since $M = M_1 \cap M_2$ and $S = S_1 \cap S_2$, if the limit is defined, then $\sup M = \inf S$, and so $\min\{\sup M_1, \sup M_2\} = \max\{\inf S_1, S_2\}$. It is straightforward case check to show that $\sup M_1 = \inf S_1 = \sup M_2 = \inf S_2$. One case check is as follows (the rest are nearly identical). Suppose that $\min\{\sup M_1, \sup M_2\} = \sup M_1$. Then $\sup M_1 \leq \sup M_2$. Since $\sup M_1 \leq \inf S_1$ and $\sup M_2 \leq \inf S_2$, we must have that $\sup M_1 \leq \inf S_1 \leq \inf S_2$, so $\inf S_2 = \max\{\inf S_1, \inf S_2\}$, whence $\inf S_1 = \inf S_2 = \sup M_1 = \sup M_2$.

The result is immediate in the converse direction. □

Theorem 17. *Let* f *be a surreal-valued function defined on* $(b, a) \cup (a, c)$. *If* $\mathcal{L}(f)[a] = l \in \text{No}$, *then for every surreal* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $0 < |y - a| < \delta$, *we have* $|f(y) - l| < \epsilon$. *Conversely, if* $l \in \text{No}$ *is a such that for every surreal* $\epsilon > 0$, *there is a* $\delta > 0$ *such that whenever* $0 < |y - a| < \delta$, *we have* $|f(y) - l| < \epsilon$, *then* $\mathcal{L}(f)[a] = l$.

Proof. Although the proof of this result is omitted in [3], it is included here for completeness.

The main idea is to adapt the proof of Theorem 22 from [3], where the authors prove that their definition of limit for convergent On-length sequences is compatible with the standard notion of convergent infinite sequences. This proof required a result regarding convergence on the tail of a sequence, while our result will use Lemma 3.

For the forward direction, first suppose that $\mathcal{L}(f)[a] = l \in \text{No}$. We must prove for every $\epsilon > 0$, there exists $\delta > 0$ so that for all $0 < |y - a| < \delta$, we have $|f(y) - l| < \epsilon$.

First, we fix a surreal $\epsilon > 0$. Since $\mathcal{L}(f)[a] = \mathfrak{l} \in \mathrm{No}$, we have

$$\left(\left\{p\colon p < \sup\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{L}_{f(y)} \cap \bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{L}_{f(y)}\right)\right\} \middle| \left\{q\colon q > \inf\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{R}_{f(y)} \cap \bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{R}_{f(y)}\right)\right\}\right) = \mathfrak{l}$$

and by Lemma 4, we have

$$M_1 = \sup\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{L}_{f(y)}\right) = \mathfrak{l}$$

$$M_2 = \sup\left(\bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{L}_{f(y)}\right) = \mathfrak{l}$$

$$S_1 = \inf\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{R}_{f(y)}\right) = \mathfrak{l}$$

and

$$S_2 = \inf\left(\bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{R}_{f(y)}\right) = \mathfrak{l},$$

so in particular,

$$M := \min\{M_1, M_2\} = \mathfrak{l} = \max\{S_1, S_2\} =: S.$$

If we are to suppose towards a contradiction that there is no δ^+ such that for all $y \in [\delta^+, a) \cup (a, a+\delta^+]$, $f(y) - \mathfrak{l} < \epsilon$, then we have for arbitrarily many $y_\alpha \in (a, c)$ that $f(y_\alpha) - \mathfrak{l} \geq \epsilon$, whence we can form a sequence $\mathfrak{Y}$ where $y_\alpha \to a$ such that for each $\alpha \in \mathrm{On}$, $f(y_\alpha) \geq \mathfrak{l} + \epsilon$. But then we have $S = \inf\left(\bigcup_{x\in(a-\delta^+,a)}\bigcap_{y\in[x,a)}\mathfrak{R}_{f(y)} \cap \bigcup_{x\in(a,a+\delta^+)}\bigcap_{y\in(a,x]}\mathfrak{R}_{f(y)}\right) \geq \mathfrak{l} + \epsilon$. Contradiction.

Similarly for the Left options, if we suppose there is no such $\delta^- > 0$ so that for for all $y \in [a - \delta^-, a) \cup (a, a + \delta^-]$, $-\epsilon < f(y) - \mathfrak{l}$, we can construct an On-length sequence $\mathfrak{Y}'$ approaching a such that $\mathfrak{l} - \epsilon \leq f(y_\alpha)$. But then, we have $M \geq \mathfrak{l} - \epsilon$. Contradiction.

We then take $\delta = \min\{\delta^+, \delta^-\}$, whence we have for all surreal $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - \mathfrak{l}| < \epsilon$.

In the converse direction, we first suppose there is a $\mathfrak{l} \in \mathrm{No}$ such that for all $\epsilon > 0$ there is some $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - \mathfrak{l}| < \epsilon$.

If we suppose towards a contradiction that $\mathcal{L}(f)[a] \neq \mathfrak{l}$, we must consider the following two cases: (i) $\mathcal{L}(f)[a]$ is not a Dedekind representation, (ii) $\mathcal{L}(f)[a]$ describes a gap $\mathfrak{g}$.

In the first case, if the limit is undefined because it fails to be a Dedekinds representation, then we would have for some $\epsilon > 0$,

$$\inf\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{R}_{f(y)} \cap \bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{R}_{f(y)}\right) - \sup\left(\bigcup_{x\in(b,a)}\bigcap_{y\in[x,a)}\mathfrak{L}_{f(y)} \cap \bigcup_{x\in(a,c)}\bigcap_{y\in(a,x]}\mathfrak{L}_{f(y)}\right) > \epsilon.$$

However, we find this is impossible by our assumption that f us a surreal-valued function on $(b, a) \cup (a, c)$ and $\mathfrak{l} \in \mathrm{No}$ such that for every surreal $\epsilon' > 0$, there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - \mathfrak{l}| < \epsilon'$. Specifically, take $\epsilon' = \frac{\epsilon}{2}$. Then by our assumption, for all $x \in (a - \delta, a)$ and $y \in (a, a + \delta)$, we have

$$|f(x) - \mathfrak{l}| < \frac{\epsilon}{2}$$

and

$$|f(x) - \mathfrak{l}| < \frac{\epsilon}{2}$$

so that by the triangle inequality, we have

$$|f(x) - f(y)| < \epsilon.$$

Contradiction.

So now consider the second case, i.e. suppose towards a contradiction that $\mathcal{L}(f)[a]$ is a gap.

We know by hypothesis that for all $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - \mathfrak{l}| < \epsilon$.

Let's first suppose that we can pick $\epsilon > 0$ such that $\mathfrak{l} - \epsilon > \mathcal{L}(f)[a]$. It follows that for $x \in B_\delta(a)\backslash\{a\}$, we have $f(x) > \mathfrak{l} - \epsilon > \mathcal{L}(f)[a]$, which is a contradiction. Hence $\mathcal{L}(f)[a] \not< \mathfrak{l}$. We can do a similar argument for $\mathfrak{l} < \mathcal{L}(f)[a]$. But then $\mathcal{L}(f)[a] = \mathfrak{l}$, which contradicts our assumption that $\mathcal{L}(f)[a]$ was a gap. This also will suffice to show that $\mathcal{L}(f)[a] = \mathfrak{l} \in \mathrm{No}$. □

Proposition 6. *For all* $a, b, c \in \mathrm{No}$ *such that* $b < a < c$ *and* $f : (b, a) \cup (a, c) \to \mathrm{No}$*, if* $\mathcal{L}(f)[a] \in \mathrm{No}^{\mathfrak{D}}$*, then* $\mathcal{L}(f)[a]$ *is not a Type II gap.*

Proof. The proof of this proposition is analogous to the proof of Lemma 26 from [3], namely, the proof that Cauchy sequences do not converge to Type II gaps.

Suppose towards a contradiction that $\mathcal{L}(f)[a] \in \mathrm{No}\mathfrak{D}$ converges to a Type II gap. By Theorem 17, it follows that we can make $|f(x)^{\mathfrak{R}} - f(y)^{\mathfrak{L}}|$ arbitrarily close to 0 where $x, y \in (a - \delta, a + \delta)\backslash\{a\}$

for sufficiently small δ. It follows that $|f(x)^{\mathfrak{R}} - \mathfrak{l}|$ can be made arbitrarily close to 0 as $\delta \to 0$. If $\mathfrak{l}$ is a type II gap, then it has normal form

$$\sum_{\alpha} \omega^{y_i} r_i \oplus (\pm\omega^{\Theta})$$

for some gap $\Theta \in \mathrm{No}^{\mathfrak{D}}$. Since $\sum_{\alpha} \omega^{y_i} r_i$ is a number, so we have

$$|g - f(y)^{\mathfrak{R}}| = |h \oplus (\omega^{\Theta})|$$

for some $h \in \mathrm{No}$. In turn, $|h \oplus (\pm\omega^{\Theta})|$ is also a Type II gap which we can make smaller than any surreal $\epsilon > 0$. Pick $\epsilon = \omega^r$ for some $r < \Theta$ (which can be done since $\Theta > \mathrm{Off}$, as ω^{Θ} is not a gap permitted in our construction of the Dedekind completion). Then either we have $h > \omega^{\Theta} > h - \omega^r$, or $h < \omega^{\Theta} < h + \omega^r$. In teh first case, let z denote the leader of the normal form of h. Since $z > \Theta$ and $z > r$, the largest power in $h - \omega^r$ is also z. But then $h - \omega^r > \omega^{\Theta}$, which is a contradiction. The argument for the second case is similar; let z denote the leader of h. But then $z < \Theta$, so the leader of $h + \omega^r$ is either $\max\{z, r\}$, implying that $h + \omega^r < \omega^{\Theta}$. Another contradiction.

Thus if $\mathcal{L}(f)[a] \in \mathrm{No}^{\mathfrak{D}}$, then $\mathcal{L}(f)[a]$ does not converge to a Type II gap. □

Proposition 7. *For all* $a, b, c \in \mathrm{No}$ *such that* $b < a < c$ *and* $f : (b, a) \cup (a, c) \to \mathrm{No}$*, if* $\mathcal{L}(f)[a] \in \mathrm{No}^{\mathfrak{D}} \backslash \mathrm{No}$*, then* $\mathcal{L}(f)[a]$ *is a gap of Type Ia, i.e* $\mathcal{L}(f)[a] = \sum_{\mathrm{On}} \omega^{y_i} r_i$ *such that* $\lim_{i \in \mathrm{On}} y_i = \mathrm{Off}$.

Proof. First, suppose that $b < a < c$ and f is as above. Further, suppose that $\mathcal{L}(f)[a] \in \mathrm{No}^{\mathfrak{D}} \backslash \mathrm{No}$. By Proposition 6, if $\mathcal{L}(f)[a]$ is not a surreal number, then it must be a gap of Type I. From here, we adapt the proof of Lemma 27 from [3] to show that $\mathcal{L}(f)[a]$ is a gap of Type Ia.

Suppose not. Then $(y_i)_{i \in \mathrm{On}}$ is a decreasing sequence approaching some number bounded below by a number $b \in \mathrm{No}$. By Theorem 17 we can choose x, y arbitrarily close to a with so that $|f(x)^{\mathfrak{R}} - f(y)^{\mathfrak{L}}|$ is arbitrarily closer to 0, which in turn means that $|f(x) - \sum_{i \in \mathrm{On}} \omega^{y_i} r_i|$ can be made arbitrarily close to 0 when δ is arbitrarily close to 0. Specifically, we can choose an x close to a so that $f(x) > \mathcal{L}(f)[a]$, i.e. $f(x)^{\mathfrak{R}}$ with $|f(x)^{\mathfrak{R}} - \sum_{i \in \mathrm{On}} \omega^{y_i} r_i| < \omega^b$. From this, it follows that $f(x) > \sum \omega^{y_i} r_i > f(x) - \omega^b$, and thus the leader of the normal form of $f(x) - \sum \omega^{y_i} r_i$ must be some $z \geq y_\alpha$ for $\alpha \in \mathrm{On}$. But this requires $z > b$, and so $f(x) - \sum_{i \in \mathrm{On}} \omega^{y_i} r_i > \omega^b$. Contradiction.

We run a similar argument for $\sum_{\mathrm{On}} \omega^{y_i} r_i > f(x)$, deriving another contradiction, as the leader of the difference of the gap and $f(x)$ will be some $z > b$.

□

The definition of limit requires that the Left and Right options are drawn from both intervals to the left and right. We now see that limits from the left and right behave as we expect them to behave. Furthermore, the limit laws are satisfied by this definition:

Theorem 18. *For functions* $f, g : (b, a) \cup (a, c) \to \mathrm{No}$*, such that* $\mathcal{L}(f)[a] = \mathfrak{l} \in \mathrm{No}$ *and* $\mathcal{L}(g)[a] \in \mathfrak{g} \in \mathrm{No}$*, we have*

Addition $\mathcal{L}(f+g)[a] = \mathcal{L}(f)[a] + \mathcal{L}(g)[a]$;

Multiplication $\mathcal{L}(fg)[a] = \mathcal{L}(f)[a]\mathcal{L}(g)[a]$.

Scalar *for* $k \in \mathbf{No}$, *for* $\mathcal{L}(f)[a] = \mathfrak{l} \in \mathbf{No}^{\mathfrak{D}}$, *then* $\mathcal{L}(kf)[a] = k\mathfrak{l}$.

Difference $\mathcal{L}(f-g)[a] = \mathcal{L}(f)[a] - \mathcal{L}(g)[a]$.

Reciprocal *If* $\mathcal{L}(f)[a] \in \mathbf{No}^{\times}$, *then* $\mathcal{L}(1/f)[a] \in \mathbf{No}^{\times}$.

Quotient *If* $\mathcal{L}(f)[a], \mathcal{L}(g)[a] \in \mathbf{No}$ *and* $\mathcal{L}(g)[a] \neq 0$, *then* $\mathcal{L}(f/g)[a] = \mathcal{L}(f)[a]/\mathcal{L}(g)[a]$.

Composition *If* $\mathcal{L}(g)[a] = \mathfrak{g} \in \mathbf{No}$, *and* $\mathcal{L}(f)[\mathfrak{g}] = f(\mathfrak{g})$, *then* $\mathcal{L}(f \circ g)[a] = f(\mathfrak{g})$.

Proof. Suppose that f, g and $b < a < c$ are as in the statement of the theorem. Further suppose that $\mathcal{L}(f)[a] = \mathfrak{l}$ and $\mathcal{L}(f)[a] = \mathfrak{g}$ where both limits are defined. We use the triangle inequality and Theorem 17 as in the standard proof found for elementary real analysis. Specifically, we fix a surreal $\epsilon_0 > 0$, and let $\epsilon_1 = \frac{\epsilon_0}{2}$. By Theorem 17, we have that there exists $\delta_1, \delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - \mathfrak{l}| < \epsilon_1$$

and

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - \mathfrak{g}| < \epsilon_1.$$

Setting $\delta_0 = \min\{\delta_1, \delta_2\}$, it follows that

$$0 < |x - a| < \delta_0 \Rightarrow |f(x) - \mathfrak{l} < \epsilon_1 \wedge |g(x) - \mathfrak{g}| < \epsilon_1$$

So

$$|f(x) - \mathfrak{l}| + |g(x) - \mathfrak{g}| < \epsilon_0.$$

Thus by the triangle inequality

$$|(f(x) + g(x)) - (\mathfrak{l} + \mathfrak{g})| < \epsilon_0.$$

Since our choices of a, b, c, f, g, and $\epsilon_0 > 0$ was arbitrary, we find that

$$\mathcal{L}(f + g)[a] = \mathfrak{l} + \mathfrak{g} = \mathcal{L}(f)[a] + \mathcal{L}(g)[a].$$

The proof for multiplication similarly follows the ordinary proof from real analysis:

Given some $\epsilon > 0$, with $\mathcal{L}(f)[a] = \mathfrak{l}$ and $\mathcal{L}(g)[a] = \mathfrak{g}$, both converging to surreal numbers, we first set $\epsilon_1 = \sqrt{\frac{\epsilon}{3}}$, $\epsilon_2 = \frac{1}{3(1+|\mathfrak{g}|)}$, and $\epsilon_3 = \frac{1}{3(1+|\mathfrak{l}|)}$. In turn, by Theorem 17, let $\delta_1 > 0$ be such that for $0 < |x - a| < \delta_1$, we have $|f(x) - \mathfrak{l}| < \epsilon_1$. Similarly, let $\delta_2 > 0$ such that for $x \in B_{\delta_2}(a)\backslash\{a\}$, $|f(x) - \mathfrak{l}| < \epsilon_2$ (and δ_3 pair for ϵ_3, and finally δ_4 such that for $x \in B_{\delta_4}(a)\backslash\{a\}$, $|g(x) - \mathfrak{g}| < \epsilon_1$).

Then we set $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Since

$$|f(x)g(x) - \mathfrak{l}\mathfrak{g}| = |\mathfrak{l}(g(x) - \mathfrak{g}) + \mathfrak{g}(f(x) - \mathfrak{l}) + (f(x) - \mathfrak{l})(g(x) - \mathfrak{g})|$$

by the Triangle inequality, we have

$$|f(x)g(x) - \mathfrak{l}\mathfrak{g}| \leq |\mathfrak{l}||g(x) - \mathfrak{g}| + |\mathfrak{g}||f(x) - \mathfrak{L}| + |f(x) - \mathfrak{l}||g(x) - \mathfrak{g}|.$$

By our choice of δ, we then have

$$|f(x)g(x) - \mathfrak{l}\mathfrak{g}| < |\mathfrak{l}|\epsilon_3 + |\mathfrak{g}|\epsilon_2 + \epsilon_1\epsilon_1 < \epsilon,$$

as desired. Thus $\mathcal{L}(fg)[a] = \mathcal{L}(f)[a]\mathcal{L}(g)[a]$

The proof for scalar multiplication follows by the proof of multiplication, where we can treat the scalar as a constant function. The proof of the difference follows by the scalar result.

The proof for non-zero reciprocals is again the standard $\delta\epsilon$ argument. If $\mathcal{L}(f)[a] = \mathfrak{l} \neq 0$, then given $\epsilon > 0$, set $\epsilon_1 = \frac{|\mathfrak{l}|}{2}$ and $\epsilon_2 = \frac{\mathfrak{l}^2\epsilon}{2}$. Choose δ_1 and δ_2 with respect to ϵ_1 and ϵ_2, and then set $\delta = \min\{\delta_1, \delta_2\}$. By the triangle inequality and some routine algebraic manipulation, we have

$$|\frac{1}{f(x)} - \frac{1}{\mathfrak{l}}| = \frac{1}{|\mathfrak{l}}\frac{1}{|f(x)|}|f(x) - \mathfrak{l}| < \frac{1}{|\mathfrak{l}|}\frac{2}{|\mathfrak{l}|}\epsilon_2 < \epsilon.$$

As a direct result of the product and reciprocal law, we get the quotient law.

Finally, for the composition law, we also apply the standard $\delta\epsilon$ proof, again provided for completeness.

Fix $\epsilon > 0$, and let $\delta_1 > 0$ so that for $0 < |y - \mathfrak{g}| < \delta_1$ so that $|f(y) - f(\mathfrak{g})| < \epsilon$. Then set $\epsilon_1 = \delta_1$, and let $\delta_2 > 0$ so that for $0 < |x - a| < \delta_2$, we have $|g(x) - \mathfrak{g}| < \epsilon_1$. By setting $y = g(x)$, we have for all $0 < |x - a| < \delta_2$ that $|f(g(x)) - f(\mathfrak{g})| < \epsilon$ as desired. □

The following theorem summarizes several key topological and analytical results of [3], including the Intermediate value theorem.

Theorem 19. *1. Every convex Class* $T \subset \mathbf{No}$ *is connected.*

2. *If* f *is continuous on* $[\mathsf{a},\mathsf{b}]$*, then* $\mathsf{f}([\mathsf{a},\mathsf{b}])$ *is connected.*

<u>*IVT*</u> *If* f *is continuous on* $[\mathsf{a},\mathsf{b}] \subseteq \mathrm{No}$*, then for every* $\mathsf{u} \in \mathrm{No}$ *that lies between* $\mathsf{f}(\mathsf{a})$ *and* $\mathsf{f}(\mathsf{b})$*, there exists a number* $\mathsf{p} \in [\mathsf{a},\mathsf{b}]$ *such that* $\mathsf{f}(\mathsf{p}) = \mathsf{u}$.

3. *A continuous function* f *has a 0 at a gap* g *if and only if for every surreal* $\epsilon > 0$*, there exists a zero of* f *in some open interval of width* ϵ *containing* g*. That is, by the IVT, continuous functions do not have isolated zeroes at gaps of* No*.*

2.5 <u>Sign Sequences and Reductions</u>

The following results are a summary of Gonshor Chapter 5 [1], as well as some results of Ehrlich & van den Dries [17], and Kuhlmann & Matusinski [27]. While each author has their own preferred notation for concatenation and representing the sign sequence, I have opted to use notation keeping in line with Kunen [28], Jech [29], and other more set-theoretically inclined authors [30].

Definition 44. *Following Gonshor, a surreal number* a *can be regarded as a function from some ordinal* α *to a set of cardinality 2. These functions are so that for two surreal numbers* a,b*, we may* ***concatenate*** *them to form a third number,* $\mathsf{a} \frown \mathsf{b}$*. The concatenation operation respects standard results on ordinal length, i.e.*

$$\iota(\mathsf{a} \frown \mathsf{b}) = \iota(\mathsf{a}) \oplus \iota(\mathsf{b})$$

as can be verified by an induction argument on the lengths of numbers.

Whenever necessary, we will use $\oplus$ to indicate ordinal addition, and $\otimes$ to indicate ordinal multiplication. Otherwise, from context $\alpha\beta$ and other string concatenations of Greek letters will indicate surreal multiplication.

Notation 4. *Given a surreal number a, we will denote by (a) the sign sequence of* a*, and write out the sign sequence as a sequence of ordered pairs* $(\langle \alpha_\mu, \beta_\mu \rangle : \mu \in \phi a)$*, where* $\phi a \in \mathrm{On}$ *denotes the number of pairs of signs in a. Note, if* $\alpha_\mu = 0$*, then* $\mu = 0$ *or* $\mu \in \mathrm{Lim}(\mathrm{On})$*, and if* $\beta_\mu = 0$*, then* μ *is the maximum element in* ϕa*, i.e. the sequence terminates at* μ*.*

Definition 45. *Given* $a \in \mathrm{No}$*, let* a^+ *denote the total number of* $+$ *appearing in the sign sequence of* a*, so*

$$a^+ = \bigoplus_{\mu} \alpha_\mu$$

as an ordinal sum.

Given $a \in \mathrm{No}_{>0}$*, define* $a^\flat$ *to be the surreal number attained by omitting the first* $\oplus$ *sign.*

Given $a \in \mathrm{No}_{<0}$*, define* $a^\sharp$ *to be the surreal number attained by omitting the first* $\ominus$ *sign.*

Given a surreal number $a = \sum_{i \in \nu a} \omega^{a_i} r_i$ *in normal form, we define the* ***reduced sequence*** $(a_i^o | i \in \nu a)$ *by omitting* $\ominus$ *from the following sign sequences:*

- *given* $\gamma \in \mathrm{On}$*, if* $a_i(\gamma) = \ominus$ *and there exists* $j < i$ *such that* $a_j(\delta) = a_i(\delta)$ *for all* $\delta \leq \gamma$*, then omit the* γ^{th} $\ominus$*;*

- *if* i *is a successor,* $a_{i-1} \frown \ominus \sqsubset a_i$ *and if* r_{i-1} *is not a dyadic rational, then omit* $\ominus$ *after* a_{i-1} *in* a_i*.*

The following theorems provide a concise overview of the sign sequence lemma, as well as the sign sequence of generalized epsilon numbers, and the proofs found in [17].

Theorem 20. *Given* $a = (\langle \alpha_i, \beta_i \rangle)_{i \in \phi a}$*, and for any* $\mu \in \phi a$*, we we have*

$$\gamma_\mu := \bigoplus_{\lambda \leq \mu} \alpha_\lambda,$$

then the sign sequence of (ω^a) *is given by*

$$(\omega^a) = \frown_{i \in \phi a} \langle \omega^{\gamma_i}, \omega^{\gamma_i + 1} \beta_i \rangle$$

Theorem 21. *Given a positive real* r *with sign sequence* $(\langle \rho_i, \sigma_i \rangle)$*, the sign sequence of* $\omega^a r$ *is*

$$(\omega^a) \frown \langle \omega^{a^+} \rho_0^\flat, \omega^{a^+} \sigma_0 \rangle \frown (\langle \omega^{a^+} \rho_i, \omega^{a^+} \sigma_i \rangle : 0 < i \leq \iota r)$$

with $\omega^{a^+} \rho$ *and* $\omega^{a^+} \sigma$ *being the standard ordinal multiplication (with absorption).*

If r *is a negative real, we reverse the signs.*

Theorem 22. *Given* $a = \sum_{i \in \nu a} \omega^{a_i} r_i$*,*

$$(a) = \frown_{i \in \nu a} (\omega^{a_i^o} r_i)$$

Corollary 1. *For all* $a \in \mathrm{No}$, *with Conway normal form* $\sum_{i \in \nu a} \omega(a_i) r_i$, *we have*

$$\iota(a) = \bigoplus_{i \in \nu} \iota(\omega(a_i^o) r_i)$$

Proof. This follows directly from $\iota(a \frown b) = \iota(a) \oplus \iota(b)$, and by induction on νa. □

The following theorem is a combination of theorems 9.5 and 9.6 in [1]

Theorem 23. *We say that* $a \in \mathrm{No}$ *is an (general)* ***epsilon number*** *if* $\omega^a = a$. *Further,*

1. $a = \frown_{\phi a} \langle \alpha_i, \beta_i \rangle$ *is an epsilon number if and only if* $\alpha_0 \neq 0$, *all* $\alpha_\mu \neq 0$ *are ordinary epsilon numbers such that* $\alpha_\mu > \sup_{\lambda \in \mu} \alpha_\lambda$, *and* β_μ *is a multiple of* $\omega^{\alpha_\mu \omega}$ *for all* $\alpha_\mu \neq 0$, *and a multiple of* $\omega^{\gamma_\mu \omega}$ *for* $\gamma_\mu = \bigoplus_{i \leq \mu} \alpha_i$ *whenever* $\alpha_\mu = 0$.
2. *For all* $\mu \in \phi a$, *we have* $\alpha_\mu(\varepsilon(a)) = \varepsilon_{\gamma_\mu(a)}$ *and* $\beta_\mu(\varepsilon(a)) = (\varepsilon_{\gamma_\mu(a)})^\omega \beta_\mu(a)$.

In fact, Gonshor proves a remarkable result that can extend to fixed points for arbitrary functions:

Theorem 24. *Suppose* $f : \mathrm{No} \to \mathrm{No}$ *and* $g, h : \mathrm{On} \to \mathrm{On}$ *such that* g *is strictly increasing continuous with image in the class of gamma numbers and* h *is arbitrary except that it never takes on the value 0. Further, assume that the sign sequence for* f *is given as follows:*

1. $\alpha_\mu(f(a))$ *has length* $g(\gamma_\mu(a))$.
2. $\beta_\mu(f(a))$ *has length* $h(\gamma_\mu(a))\beta_\mu(a)$

Then $f(a) = a$ *if and only if for all* $\mu \in \mathrm{On}$,

1. $g(\alpha_\mu(a)) = \alpha_\mu(a)$;

2. $\alpha_\mu > \sup_{i \in \mu} \alpha_i(a)$;

3. $\beta_\mu(a)$ *is a multiple of* $h(\gamma_\mu(a))^\omega$.

The following is an immediate corollary relevant to Veblen functions (see Chapter 5.1 for a definition):

Corollary 2. *Let* $(\varphi_\alpha)_{\alpha \in \mathrm{On}}$ *denote the Veblen hierarchy. Then for any* $a \in \mathrm{No}$*, the sign sequence of* $(\varphi_\alpha(a))$ *is given as follows:*

1. $(\varphi_0(a)) = (\omega(a))$;

2. $(\varphi_1(a)) = (\varepsilon(a))$;

3. *for* $\delta \in \mathrm{On}_{>1}$*, then*

$$(\varphi_\delta(a)) = \frown_{\mu \in \phi a} \langle \varphi_\delta(\gamma_\mu(a)), (\varphi_\delta(\gamma_\mu(a)))^{\omega(\delta)} \beta_\mu(a) \rangle$$

Proof. This follows by induction on $\delta \geq 1$, the definition of the Veblen hierarchy, and Theorem 24 where $f = g = h = \varphi_\delta$. It is immediate that $\varphi_\delta(\gamma_\mu(a))^\omega$ factors through $\varphi_\delta(\gamma_\mu(a))^{\omega(\delta)}$ whenever $\delta \geq 1$. □

Finally, the following results from Chapter 6 of [1] and [17].

Fact 4. *Supposing that* $\iota(a) \leq \iota(b) \leq \iota(c)$*:*

1. $\iota(a + b) \leq \iota(a) + \iota(b)$;

2. $\iota(ab) \leq 3^{\iota(a)+\iota(b)}$;

3. $|\iota(a^{-1})| \leq \aleph_0 |\iota(a)|$;

4. *For* $a \in \text{No} \backslash \mathbb{D}$, *then* $|\iota(\omega(a))| = |\iota(a)|$;

5. *for any non-zero real r and* a, $|\iota(\omega(a)) \cdot r| = |\iota(\omega(a))|$;

6. *If* $\omega(b)r$ *is a term in the normal from of* a, *then* $\iota(\omega(b)) \leq \iota(a)$;

7. *For* $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$, *then* $|\beta| \leq |lub_{\alpha\in\beta}[\iota(a_\alpha)\omega]|$. *(This result refers to the least upper bound of ordinals on the right hand side, and cardinalities on the left hand side).*

8. *For* $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$, *then* $|\iota(a)| \leq |lub_{\alpha\in\beta}\iota(a_\alpha), \omega|$;

9. *For* $a = \sum_{\alpha \in \beta} \omega(a_\alpha) r_\alpha$ *and* $lub_{\alpha\in\beta}(|\beta|, |\iota(a_\alpha)|, \aleph_0) \leq \kappa$, *then* $|\iota(a)| \leq \kappa$.

10. *The set of surreals with lengths less than a fixed ordinal* ϵ *number form a subfield of surreal numbers;*

11. *For* $a_1, \ldots, a_n$ *arbitrary surreal numbers and* $r_1, \ldots, r_n$ *rational numbers, then* $|\iota(\sum r_i a_i)| \leq |\max \iota(a_i)| \aleph_0$.

12. *An ordinal upperbound for the cardinality of* κ *will be the least* ϵ *number larger than* α.

13. *The subset of surreal numbers* $\{a \mid |\iota(a)| \leq \kappa\}$ *for any fixed infinite cardinal* κ *will form a real closed field. Furthermore, since all operations will depend on finitely many elements of the condition* $\iota(a) \leq d$, *we may strengthen this to* $\iota(a) < d$.

14. *For dyadic rationals* $a > 0$, $\iota(a) = \iota([a]) + \iota(a - [a])$ *where* $[a]$ *denotes the natural number part of* a;

15. *For* $a, b \in \mathbb{R}$, $\iota(ab) \leq \iota(a)\iota(b)$*;*

16. *Let* $x, y \in \text{No}$*, and* $0 < r \in \mathbb{R}$*, then we have:*

    (a) $(x+y)^+ \leq x^+ + y^+$*;*

    (b) $\iota(\omega^x r) = \iota(\omega^x) \oplus \omega^{x^+} \otimes \iota(r^\flat)$

    (c) *if* r *is a dyadic rational, then $ω(ω^x r)=ι(ω^x) +* $\omega^{x^+}\iota(r^\flat)$*;*

    (d) *if r is not a dyadic rational, then* $\iota(\omega^x r) = \iota(\omega^x) + \omega^{x^+}(\omega - m)$ *where* $m \in \omega$ *is the coefficient of* ω^{x^+} *in the Cantor normal form of* $\iota(\omega(x))$.

17. *For all surreal numbers x and y such that* $\iota(\omega^x \omega^y) \leq \iota(\omega^x)\iota(\omega^y)$*, then for all* $r, s \in \mathbb{R}$*,*

$$\iota((\omega^x r)(\omega^y s)) \leq \iota(\omega^x r)\iota(\omega^y s)$$

18. *If* $a = \omega^x r$ *and* $b = \omega^y s$*, then* $\iota(ab) \leq \iota(a)\iota(b)$.

19. *For all* a, $\iota(a) \leq \iota(\omega^a) \leq \omega^{\iota(a)}$*;*

20. *For all* a, $\nu a \leq \iota a$*;*

21. *For all* $\alpha \in \nu a$, $\iota(\omega^{a_\alpha} r_\alpha) \leq \iota(a)$*;*

22. *If* $\xi \in \text{On}$ *such that* $\iota(\omega^{a_\alpha} r_\alpha) \leq \xi$ *for all* $\alpha \in \nu(a)$*, then* $\iota(a) \leq \xi \nu(a)$.

23. *For any surreal numbers a and b,* $\iota(ab) \leq \omega \iota(a)^2 \iota(b)^2$

Reductions were first introduced in [1] as part of the proof the sign sequence theorem.[1]

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[1]See Ch. 5 Chapter E of [1] for further details. Specifically the sign sequence theorem is Theorem 5.12.

In [1], the reduction of a surreal number is denoted by $\mathfrak{a}^{o}$. Throughout this dissertation, we will denote by $\mathfrak{b}^{o}\mathfrak{a}$ that $\mathfrak{b}$ has been reduced relative to a specific $\mathfrak{a}$. In practice, reduction can be done with respect to multiple numbers, but given that the reduction requires agreement on a common head, reductions of the first type are to be carried out with respect to a previous $\mathfrak{a}$ sharing the largest common head with b, whereas reductions of the second type are only be carried out with respect to the immediate predecessor of $\mathfrak{b}$, in which case the notation of $\mathfrak{b}^{o}$ may be used if it is clear in the immediate context that we're considering reduction with respect to the ordinal head of $\mathfrak{b}$.,

From this definition of reductions internal to a given sequence of descending surreal numbers, we define a binary relation $\multimap$ as follows:

Definition 46. *Let* $(\mathfrak{a}, \mathfrak{r})$ *and* $(\mathfrak{b}, \mathfrak{s})$ *be elements of* $(\mathrm{No} \times \mathbb{R}^{\times})\backslash(\mathrm{On} \times \mathbb{D})$. *We say* $(\mathfrak{b}, \mathfrak{s})$ *is* ***reduced relative to*** $(\mathfrak{a}, \mathfrak{r})$, *and denote this*[1] *by* $(\mathfrak{b}, \mathfrak{s}) \multimap (\mathfrak{a}, \mathfrak{r})$, *if* $\mathfrak{a} > \mathfrak{b}$ *and either of the following conditions hold:*

1.

$$\exists x \in \mathrm{On} \forall y \leq x (\mathfrak{b}(x) = \ominus \wedge \mathfrak{a}(y) = \mathfrak{b}(y))$$

2.

$$((\mathfrak{a} \frown \ominus) \sqsubseteq \mathfrak{b}) \wedge \mathfrak{r} \in \mathbb{R}\backslash\mathbb{D}$$

[1] *Throughout the dissertation we will use* $(\mathfrak{b}, \mathfrak{s})^{o}(\mathfrak{a}, \mathfrak{r})$ *and* $\mathfrak{b}^{o}\mathfrak{a}$ *to denote the surreal number formed by reduction, in contrast to the relation that one number is reduced relative to another number.*

2.5.0.1 $\multimap$ is a strict order relation

We immediately can see the following hold:

Theorem 25. $\multimap$ *is irreflexive.*

Proof. For any $(a, r) \in \mathrm{No}^{\times} \times \mathbb{R}^{\times}$, since $>$ is irreflexive, we have that $\multimap$ must be irreflexive. □

Theorem 26. $\multimap$ *is transitive.*

Proof. Suppose that $(b, s) \multimap (a, r)$ and $(c, t) \multimap (b, s)$. Then we have

- $a > b > c$;
- conditions 1 or 2 holding for (a, b) and (b, c).

We consider four cases now. If condition 1 holds for both pairs, then $\exists x_{ba}, x_{cb}$ such that $\forall y_1 \leq x_{ba}, y_2 \leq x_{cb}$, we have that $b(y_1) = a(y_1)$, $c(y_2) = b(y_2)$, $b(x_{ba}) = a(x_{ba}) = \ominus$, and $b(x_{cb}) = c(x_{cb}) = \ominus$. But then for $x = \min\{x_{ba}, x_{cb}\}$, we have that $a(x) = \ominus$, and for all $y \leq x$, $c(x) = a(x)$. Thus $(c, t) \multimap (a, r)$.

If condition 1 holds for the first pair, and condition two holds for the second pair, then because $b \overset{\sqsubseteq}{\neq} c$, by condition 1 we have some $x \in \mathrm{On}$ such that for all $y \leq x$, $a(y) = b(y) = c(y)$ and $b(x) = c(x) = \ominus$, whence $(c, t) \multimap (a, r)$.

If condition 2 holds for the first pair, and condition 1 holds for the second pair, we have that $a \frown \ominus \sqsubseteq b$ and $\exists x \forall y \leq x(b(y) = c(y))$. In particular, either condition 1 holds for (c, t) and (a, r), or $a \frown \ominus \sqsubseteq c$ as well, since we can take x to be the least such ordinal that $c(x) = \ominus$, and either $a(x) = \ominus$ by the conditions above, or a is an ordinal and $x = \iota(a)$. Thus $(c, t) \multimap (a, r)$.

Finally, if condition 2 holds for both pairs, then we have $\mathfrak{a} \frown \ominus \sqsubseteq \mathfrak{b} \frown \ominus \sqsubseteq \mathfrak{c}$, and $r, s \in \mathbb{R}\backslash\mathbb{D}$, whence $(\mathfrak{c}, t) \multimap (\mathfrak{a}, r)$. In particular, we find that condition 1 is satisfied for $(\mathfrak{c}, t)$ and $(\mathfrak{b}, s)$ by setting $x = \iota(\mathfrak{a})$.[1]

□

Consequently, we have that:

Proposition 8. $\multimap$ *is a strict order on* $(\mathrm{No} \times \mathbb{R}^{\times})\backslash(\mathrm{On} \times \mathbb{D})$.

In fact, we can decompose $\multimap$ into two disjoint order relations as follows:

Theorem 27. *If* $\mathfrak{a} \in \mathrm{No}\backslash\mathrm{On}$*, then if* $(\mathfrak{b}, s) \multimap (\mathfrak{a}, r)$ *and condition 2 holds, then condition 1 must also hold.*

Proof. If $\mathfrak{a} \in \mathrm{No}\backslash\mathrm{On}$, then $\beta_0(\mathfrak{a}) \neq 0$, and if condition 2 holds, we must have $\mathfrak{a} \frown \ominus \sqsubseteq \mathfrak{b}$, so in particular, we must have $\mathfrak{a}(\alpha_0(\mathfrak{a})) = \mathfrak{b}(\alpha_0(\mathfrak{a})) = \ominus$, so condition 1 should also hold. □

Hence, we can decompose $\multimap := \multimap_1 \coprod \multimap_2$, where $\multimap_1$ is the component where condition one holds, and which partially orders $((\mathrm{No}\backslash\mathrm{On}) \times \mathbb{R}^{\times}) \times ((\mathrm{No}\backslash\mathrm{On}) \times \mathbb{R}^{\times})$. In particular, by the result above, we find that $\multimap_1$ applies to $\mathrm{No}\backslash\mathrm{On} \times \mathbb{R}^{\times}$, and furthermore, since the satisfaction of condition 1 is independent of our choice of r, s, we can project $\multimap_1$ down to a strict partial order on $\mathrm{No}\backslash\mathrm{On}$ without issue.

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[1] This should be revisited in the future, since condition 2 for triples $\mathfrak{a} > \mathfrak{b} > \mathfrak{c}$ can only be applied to successor pairs, and not $\mathfrak{a}$ and $\mathfrak{c}$.

On the other hand, we find that $\multimap_2$ must be defined solely on $(\text{No}\backslash\text{On} \times \mathbb{R}^{\times}) \times (\text{On} \times \mathbb{R}\backslash\mathbb{D})$. Consequently, maximal elements[1] with respect to $\multimap$ are those pairs that are the second component of $\multimap_2$. We clarify this with the following theorem.

Theorem 28. *For any arbitrary chain* $\mathfrak{C} = \langle(a_i, r_i) \mid i \in I\rangle$*, where* I *is a well-ordered class in* NGB *(so either a set or proper Class), and ordered by* $\multimap$ *such that* $a_i < a_{i+1}$*, there is always a unique minimal* $\Omega_{\mathfrak{C}} \in \text{On}$ *such that* $a_i \leq \Omega_{\mathfrak{C}}$ *for all* $i \in I$*. Furthermore, there is* $i \in I$ *such that* $a_i = \Omega_{\mathfrak{C}}$ *if and only if* $i = \max I$*. Otherwise, we can extend* $\mathfrak{C}$ *to a maximal chain by adding* $\langle\Omega, r_*\rangle$*, where* $r_* \in \mathbb{R}\backslash\mathbb{D}$ *is an arbitrary real, non-dyadic rational number.*

Proof. Let $\mathfrak{C}$ be an arbitrary chain ordered by $\multimap$. In particular, for all but the final $i \in I$, $a_i \multimap_1 a_{i+1}$ must be the case, as it must also be the case that $\alpha_0(a_i) = \alpha_0(a_j)$ for all $i, j \in I$ and $0 \leq \beta_0(a_{i+1}) \leq \beta_0(a_i) \neq 0$ for all $i, i+1 \in I$. Thus, we may set $\Omega_{\mathfrak{C}} := \alpha_0(a_0)$, as by construction, $a_i \leq \Omega_{\mathfrak{C}}$ for all $i \in I$, and if $a_i = \Omega_{\mathfrak{C}}$, the chain terminates at a_i.

We note that this is the smallest such ordinal where this is possible, since for any ordinal $\gamma \in \Omega_{\mathfrak{C}}$, since $\alpha_0(a_i) > \gamma$, $a_i > \gamma$. By minimality, it is unique.

The "furthermore" result follows immediately by the definition of $\multimap$, and the result extending $\mathfrak{C}$ follows by noting that if

$$\mathfrak{C} \frown \langle\Omega_{\mathfrak{C}}, r_*\rangle \overset{\sqsubset}{\neq} \mathfrak{D},$$

[1] Throughout this dissertation, we take an element to be maximal with respect to the ordinary linear ordering on the first component.

then $(\Omega_{\mathfrak{c}}, r_*) \multimap (d, t)$ such that $\alpha_0(d) = \Omega_{\mathfrak{c}}$. But this is absurd. □

We now find that $\multimap$ can be used to partially order No by the following theorem.

Theorem 29. *Define an equivalence relation* E *on* $\text{No}\backslash\text{On} \times \mathbb{R}^\times \coprod \text{On} \times \mathbb{R}\backslash\mathbb{D}$ *such that* $(a, r)\mathsf{E}(b, s)$ *if and only if*

$$a = b \wedge (a \in \text{On} \rightarrow r, s \in \mathbb{R}\backslash\mathbb{D}).$$

Let $\mathfrak{M} := (\text{No}\backslash\text{On} \times \mathbb{R}^\times \coprod \text{On} \times \mathbb{R}\backslash\mathbb{D})/\mathsf{E}$. *Then* $\mathfrak{M} \cong \text{No}$ *and* $\mathfrak{M}$ *inherits a partial order from* $\multimap$.

Proof. It is immediate that E defined above is an equivalence relation. Furthermore, it is immediate that every equivalence class is in one-to-one correspondence with No.

Consider $\multimap_*$ the quotient of $\multimap$ with respect to E. We check that it is a strict partial order on $\mathfrak{M}$ as follows:

Irreflexivity $(a, r) \not\multimap (a, r) \iff [(a, r)]_{\mathsf{E}} \not\multimap_* [(a, r)]_{\mathsf{E}}$;

Anti-symmetry Suppose $(a, r) \multimap (b, s)$. Then either $(a, r) \multimap_1 (b, s)$ or $(a, r) \multimap_2 (b, s)$. In either case, $a \neq b$, whence $[(b, s)]_{\mathsf{E}} \not\multimap_* [(a, r)]_{\mathsf{E}}$.

Transitivity Suppose that $(a, r) \multimap (b, s)$ and $(b, s) \multimap (c, t)$. Then $[(a, r)]_{\mathsf{E}} \multimap_* [(c, t)]_{\mathsf{E}}$ follows from $(a, r) \multimap (c, t)$, as either $(a, r) \multimap_1 (c, t)$ or $(a, r) \multimap_2 (c, t)$. If $(a, r) \multimap_1 (c, t)$, then condition 1 holds, and it is immediate that $[(a, r)]_{\mathsf{E}} \multimap_* [(c, t)]_{\mathsf{E}}$. If $(a, r) \multimap_2 (c, t)$, then $c \frown \ominus \sqsubseteq a$ and $t \in \mathbb{R}\backslash\mathbb{D}$, whence $[(a, r)] \multimap_* [(c, t)]$ holds.

Thus $\multimap_*$ is a strict partial order on $\mathfrak{M}$, and in turn, we have induced a strict partial order on No, such that the maximal elements of the order are the ordinals by Theorem 28. □

The partial order that we have induced on No is not well-behaved in the following sense:

Theorem 30. *Some chains ordered by $\multimap_1$ form proper classes.*

Proof. Consider arbitrary $(a, r) \in \mathrm{No}\backslash\mathrm{On} \times \mathbb{R}^{\times}$. Set $a_0 = a$ and let $a_{i+1} := (a_i) \frown \langle 1, 1\rangle$, and for limit ordinals λ, let $a_\lambda = (a) \frown_\lambda \langle 1, 1\rangle$. Let $\mathfrak{C} := \langle (a_i, r) \mid i \in \mathrm{On}\rangle$. It is immediate that $\mathfrak{C}$ is a proper class, as it is the size of the ordinals. □

It should be noted that while chains can be of arbitrary length, as in the construction of Theorem 30, by Theorem 28, every chain ordered by $\multimap$ can be given a maximum element, namely $(\Omega_{\mathfrak{C}}, r)$ where r is a non-dyadic rational real number. In turn, the projection of the chain into No has a unique maximum element with respect to the induced partial order. The tradeoff for uniqueness is that the reduction relationship ceases to have its original use: determining the sign sequence of a surreal number given its Conway normal form as we lose information about the coefficient of a monomial whose exponent is an ordinal number. Nonetheless, even though arbitrary chains can be very badly behaved, being that chains of arbitrary cardinality exist, we can prove several results that control the structure of reduction (see Chapter 9.1 for additional details).

2.6 Pseudo-absolute values

We review the following definition before providing several relevant pseudo-absolute values that will help establish bounds on complexity.

Definition 47. *Let $\sigma : S_1 \to S_2$ be a map between two semi-rings. We say σ is a **pseudo-absolute value if the following holds:***

1. $\sigma(x) = 0 \iff x = 0$*;*

2. $\sigma(xy) \leq \sigma(x)\sigma(y)$*;*

3. $\sigma(x + y) \leq \sigma(x) + \sigma(y)$

Pseudo-absolute values are of vital importance in the general analysis of surreal structures. It is an important open question that the length function, $\iota : \mathrm{No} \to \mathrm{On}$ is a pseudo-absolute value, as it is an open question whether $\iota(xy) \leq \iota(x)\iota(y)$ for all $x, y \in \mathrm{No}$. This is the alleged product lemma found in [2] and [1]. van den Dries and Ehrlich were able to prove a weaker inequality, $\iota(xy) \leq \omega\iota(x)^2\iota(y)^2$ for their purposes, but we will provide a sharper result using several pseudo-absolute values.

While the status of the product lemma is an open question, the following theorem can be found in both [1] and [2]:

Theorem 31. *For all* $a, b \in \mathrm{No}$*,* $\iota(a + b) \leq \iota(a) + \iota(b)$*.*

The proof for Theorem 31 is similar to the proof of the following Theorem.

Theorem 32. *For all* $a, b \in \mathrm{No}$*,* $\phi(a + b) \leq \phi(a) + \phi(b)$*.*

Proof. We induct on the lengths of a, b, noting that necessarily for all $a \in \mathrm{No}$, $\phi(a) \leq \iota(a)$.

In our base case, we have $\iota(a) = \iota(b) = \phi(a) = \phi(b) = 0$, and so immediately we have $\phi(a + b) \leq \phi(a) + \phi(b)$.

Now suppose for all pairs of $(a,b) \in \mathrm{No} \times \mathrm{No}$ such that $\iota(a) + \iota(b) < \mu$ that $\phi(a+b) \leq \phi(a) + \phi(b)$. Then for a pair (a,b) such that $\iota(a) + \iota(b) = \mu$, recall that

$$a + b = \left\{a^L + b, a + b^L\right\} \mid \left\{a^R + b, a + b^R\right\}.$$

Since $a^L + b, a + b^L, a^R + b, a + b^R \overset{\sqsubseteq}{\neq} (a+b)$, and for all proper initial segments $y \sqsubseteq x$, $\phi(y) \leq \phi(x)$, we have $\phi(a^L + b) \leq \phi(a^L) + \phi(b) \leq \phi(a) + \phi(b)$ and similarly for the other sums.[1]

In turn, for each initial segment $(a+b)^L$ in the canonical representation of $(a+b)$, by our induction hypothesis, there are some a^L, b^L, such that $\phi((a+b)^L) \leq \phi(a^L) + \phi(b) \leq \phi(a) + \phi(b)$ or $\phi((a+b)^L) \leq \phi(a) + \phi(b^L) \leq \phi(a) + \phi(b)$. To see this, for example, suppose we're looking at one of the Left options of $(a+b)$ where b is fixed, e.g. $a^L + b$. Then we may apply our induction hypothesis along the Left options of a. The same reasoning holds for $(a+b)^R$ with respect to a^R, b^R. Hence $\phi((a+b)') \leq \phi(a) + \phi(b)$ holds for all proper initial segments $(a+b)'$ of $a+b$, from which we derive $\phi(a+b) \leq \phi(a) + \phi(b)$. □

On the other hand, we can show the following straight forwardly:

Theorem 33. *For all* $a, b \in \mathrm{No}$, $\nu(a+b) \leq \nu(a) + \nu(b)$.

---

[1] Note that if $\phi(a^L) = \phi(a)$, then necessarily for such a^L, $\iota(a^L) > \iota(a^R)$ for all possible a^R, so $\phi(a^R) + 1 \leq \phi(a^L)$ for all possible a^R, and vice versa if $\phi(a^R) = \phi(a)$ for some a^R. This result is a consequence of simplicity in the sense of Ehrlich [9, 17], as the cut definition of a surreal number would require that the simplest number will add to $\alpha_{\phi a^L}(a^L)$ (respectively $\beta_{\phi a^L}(a^L)$) to form a.

Proof. Equality is immediate if a and b share no common exponents in their respective Conway normal forms. Otherwise we have strict inequality if there are shared exponents in order to avoid double counting. □

Furthermore, because of the well-ordering property, given two surreal numbers $a, b \in \text{No}$, by considering the formal sum expansion of their product, we have $\nu(ab) \leq \nu a \nu b$. From this we can conclude:

Theorem 34. $\nu : \text{No} \to \text{On}$ *is a pseudo-absolute value.*

We would like to check that $\phi : \text{No} \to \text{On}$ also forms a pseudo-absolute. However, it does not. Consider the following example. Let $a = 2\frac{1}{4}$ and $b = 1\frac{1}{8}$. Then both a and b are dyadic rationals such that $\phi(a) = \phi(b) = 1$. However, $ab = \frac{81}{32} = 2\frac{17}{32}$, so $\phi(ab) = 2$.

At the moment, we can show that

Theorem 35. *For all* $a, b \in \text{No}$*,* $\phi(ab) \leq 3\phi(a)\phi(b)$.

Proof. First, we note that for all $a' \in L_a \cup R_a$, $\phi(a') \leq \phi(a)$.

To prove this, we will induct on the lengths of a, b. Whenever, $a = 0$ or $b = 0$, then we have $\phi a = 0$ and $\phi b = 0$ respectively.

So now, supposing this result holds for all pairs $(a, b) \in \text{No} \times \text{No}$ such that $\iota(a) + \iota(b) < \mu$. Then for (a, b) such that $\iota(a) + \iota(b) = \mu$, we have for all $a' \in L_a \cup R_a$ and $b' \in L_b \cup R_b$ that $\phi(a'b') \leq \phi(a')\phi(b')$ as our induction hypothesis.

Next, we unfold the genetic definition of multiplication:

$$ab = \left\{a^L b + ab^L - a^L b^L, a^R b + ab^R - a^R b^R\right\} \mid \left\{a^L b + ab^R - a^L b^R, a^R b + ab^L - a^R b^L\right\}$$

Immediately, by Theorem 32 and our induction hypothesis we find

$$\begin{aligned} \phi(a^L b + ab^L - a^L b^L) &\leq \phi(a^L b) + \phi(ab^L) + \phi(a^L b^L) \\ &\leq \phi(a^L)\phi(b) + \phi(a)\phi(b^L) + \phi(a^L)\phi(b^L) \\ &\leq 3\phi(a)\phi(b) \end{aligned}$$

and similarly for the other terms in the genetic definition of multiplication. Thus,

$$\phi(ab) \leq 3\phi(a)\phi(b).$$

□

So we are faced with the perplexing situation where for all $a \in \mathrm{No}$ we have $\nu(a), \phi(a) \leq \iota(a)$, while ν is a pseudo-absolute value and ϕ fails to be a pseudo-absolute value.[1] Fortunately for the purposes of this dissertation, because we are considering the very coarse restriction of

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[1]Perhaps this is not so perplexing, since we can have very large ι with very small ϕ: just consider On. Each ordinal has ϕ value 1, but can be arbitrarily large! For a non-trivial instance where the product inequality fails for ϕ, consider $a = 3\frac{1}{4}$ and $b = 5\frac{1}{2}$. By sign-sequence expansion correspondence of the dyadic rationals, $\phi(a) = \phi(b)$, but $\phi(ab) = 2$, as $ab = 17\frac{7}{8}$ has the corresponding sign sequence $\langle 18, 1\rangle \frown \rangle 2, 0\rangle$

subtrees of No of a given height without further restrictions on the behavior of branches, we will not need to possess the full information about the length of a pair of surreal numbers. We will only need the leading monomial term of its length, as we will discuss in Chapter 4.2.

2.7 Real-algebraic Geometry

Before stating the Positivestellensatz, we recall several definitions from real algebraic geometry:

Definition 48. *With* R *an ordered field,* K *a real closed field containing* R *as an ordered subfield, and* $\bar{x}$ *an n-tuple of indeterminates, for a given subset* $X \subset K^n$*, the* ideal

$$\mathcal{I}(X) = \{p \in R[\bar{x}] \mid \forall \bar{x} \in X.p(\bar{x}) = 0\}$$

consists of the polynomials in the polynomial ring that vanish on X*. If given a set of polynomials* $G \subset R[\bar{x}]$*, the ideal generated by* G *is denoted by* $\langle G \rangle$*.*

Let $\mathcal{A}(X)$ *denote the semiring of polynomials in* $R[x]$ *that are nonnegative on* X*. For a given subset* $S \subset R[\bar{x}]$*, let*

$$\mathcal{V}(S) = \{\bar{x} \in K^n \mid \forall p \in S.p(\bar{x}) = 0\}.$$

Further, let

$$\mathcal{W}(S) := \{\bar{x} \in K^n \mid \forall p \in S.p(\bar{x}) \geq 0\}.$$

A subset $W \subset K^n$ *is* ***semi-algebraic*** *if* $W = \mathcal{W}(S)$ *for some finite set* $S \subset R[\bar{x}]$*.*

For any subset for any subset C *of a ring containing* R*, let* $S(C)$ *be the semiring generated by the positive elements of* R *and the square elements of* C*.*

For a commutative ring C *and an ideal* $I \subset C$*, and a subsemiring* $A \subset C$ *containing all squares in* C*, the* ***A-radical of I*** *is the subset of* C *such that*

$$\mathcal{Q}_A(I) = \{c \mid c_a^{2m} \in I, \text{ for some } m > 0, a \in A\}.$$

An ideal I is an A-radical ideal *if it is its own A-radical.*[1]

A set $P \subset R[\bar{x}]$ *is a* ***cone*** *if*

1. $f_1, f_2 \in P$ *implies that* $f_1 + f_2 \in P$*;*
2. $f_1, f_2 \in P$ *implies that* $f_1 f_2 \in P$*;*
3. $f \in R[\bar{x}]$ *implies that* $f^2 \in P$*.*

P is a ***proper cone*** *if* $-1 \notin P$*. For a given set* $S \subset R^n$*, the corresponding cone*

$$\mathcal{C}(S) := \{f \in R[\bar{x}] \mid f(\bar{x}) \geq 0 \forall x \in S\}$$

We say $f \in R[\bar{x}]$ *is a* ***sum-of-squares*** *if* $f(x) = \sum_{i=1}^{m} [s_i(x)]^2$ *for polynomials* $s_1, \ldots, s_m$*. We denote the cone of sum-of-squares polynomials with coefficients in* R *by* Σ_R*.*

Given $F = \{f_1, \ldots, f_m\} \subset R[\bar{x}]$*, we define the* ***monoid of F*** *by* $\text{MON}(F)$ *as the set of all finite products of polynomials* $f_i \in F$ *together with 1.*

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[1] *One can consult Lemma 1 from [31] to see proof that every* $\mathcal{Q}_A(I)$ *is an A-radical ideal.*

The smallest cone containing F *is given by*

$$\mathcal{C}(F) = \{\sum_{i=0}^{n} s_i g_i \mid s_i \in \Sigma_R, g_i \in \text{MON}(F)\}$$

Finally, an ideal $I \subset R[\bar{x}]$ *is* ***real*** *over* R *if for every* $\sum_{i=1}^{n} f_i^2 p_i \in I$ *with* $f_i \in R[\bar{x}]$ *and* $p_i \in R^2 \backslash \{0\}$, $f_1, \ldots, f_n \in I$.

Next, we recall the Krivine-Stengle *Positivstellensatz* and the weak form of the Positivstellensatz [31, 32]

Theorem 36. 1. *For an ordered ring* R *such that* $k = (\mathsf{Frac}(R))^{rcf}$, *and finite sets of polynomials* $F, G \subset R[\bar{x}]$, *let* $W = \mathcal{W}(F) \cap \mathcal{V}(G)$. *Define the preordering associated with* W *as the set*

$$P(F, G) := \mathcal{C}(F) + \langle G \rangle.$$

Then for any polynomial $p \in R[\bar{x}]$,

- $\forall \bar{x} \in W.\ p(\bar{x}) \geq 0$ *if and only if* $\exists q_1, q_2 \in P(F, G)$ *and* $s \in \mathbb{Z}$ *such that* $q_1 p = p^{2s} + q_2$.
- $\forall \bar{x} \in W.\ p(\bar{x}) > 0$ *if and only if* $\exists q_1, q_2 \in P(F, G)$ *such that* $q_1 p = 1 + q_2$.

Weak Positivstellensatz) For K *a real closed field, and* F, G, H *finite subsets of* $K[\bar{x}]$,

$$\{x \in K^n \mid \forall f \in F. f(x) \geq 0 \wedge \forall g \in G. g(x) = 0 \wedge \forall h(x). h(x) \neq 0\} = \emptyset$$

if and only if

$$\exists f \in \mathcal{C}(F)\exists g \in \langle G\rangle\exists n \in \mathbb{N}. f + g + (\prod H)^{2n} = 0.$$

In turn, we have the following immediate corollary:

Corollary 3. *If* R *is an ordered ring, and* $p \in R[x]$*, then* $p(x) \geq 0$ *for all* $x \in R$ *if and only if* $p = \sum s_i$ *for finitely many* $s_i \in \Sigma_R$*.*

We also have the following two Lemmas and Corollary of the Real nullstellensatz courtesy of Dickmann [33]:

Lemma 5. *If* I *is a proper ideal of* $R[\bar{x}]$*, then* I *is real over* R *if and only if* I *is a radical ideal and the intersection of finitely many prime ideals that are real over* R*.*

Lemma 6. *For any subset* $X \subset R^n$*,* $\mathcal{I}(X)$ *is real over* $K = R^{rcf}$*.*

Corollary 4. *For* k *a real closed field and* $I \subseteq k[\bar{x}]$*, then* $I = \mathcal{I}(\mathcal{V}(I))$ *if and only if* I *is real over* k*.*

Dickmann's proof of Corollary 4 is notable because it makes use of model completeness in the forward direction. Moreover, because multiplicative inverses are genetic functions, our analysis can extend beyond polynomial terms defining genetic functions to genetic functions defined by rational functions (the exponential function is the most notable example). However, because our principal interest is whether all rational function terms appearing in the Left option are non-positive, it will always suffice to consider the sign value of the polynomial terms in the rational expression of the options. Before beginning the general survey from polynomial to rational functions, we recall some definitions and the proof of Hilbert's 17th problem, which is one of many notable examples of model completeness being satisfied by substructures of the surreal numbers.

Definition 49. *Given a real-closed field* k *and a rational function* $f \in k(\bar{x})$*,* f *is* ***positive semi-definite*** *if for all* $\bar{a} \in k^n$*,* $f(\bar{a}) \geq 0$*.*

Theorem 37. *If* k *is a real closed field and* $f \in k(\bar{x})$ *is a positive semi-definite rational function, then* f *is the sum of squares of rational functions over* k*.*

Proof. Suppose that k is a real closed field and towards a contradiction that $f(\bar{x}) \in k(\bar{x})$ is a positive semi-definite rational function such that it is not the sum of squares of rational functions in $k(\bar{x})$. Since any positive cone P of $k(\bar{x})$ contains $\Sigma_{k(\bar{x})}$, consider K the real algebraic closure of $\langle k(\bar{x}), P \rangle$. Then because $f(\bar{x})$ is not the sum of squares, $K \models \exists \bar{v}.f(\bar{v}) < 0$. However, by model completeness, $k \models \exists \bar{v})f(\bar{v}) < 0$, which contradicts our assumption that $f(\bar{x})$ is positive semi-definite. □

2.8 Model Theory with Classes

Before proceeding further, we need to remark on the definability of truth in class-structures.

In the classical setting, an $\mathcal{L}$-structure $\mathcal{M}$ has an underlying set M as its universe, and $\mathcal{M}$ satisfies an $\mathcal{L}-$theory T, which we denote by $\mathcal{M} \models T$, whenever for every sentence $\phi \in T$, we have $\mathcal{M} \models \phi$. For any $\mathcal{L}$-sentence ϕ, we say that $\mathcal{M}$ *satisfies* ϕ, or that ϕ is *true* in $\mathcal{M}$. In general, the satisfaction relation is inductively defined on $\mathcal{L}$ formula with respect to the interpretation of the $\mathcal{L}$ symbols in $\mathcal{M}$ and the existence of substitution maps sending $\mathbb{N}$-indexed variables to elements in the underlying universe.

Given a sentence $\phi = Q_1 v_1 \dots Q_n v_n \varphi(\bar{v})$, where φ is a quantifier free formula, this leads to the following equivalence

$$\mathcal{M} \models \phi \iff \mathcal{M} \models \varphi(\bar{a}),$$

where $\bar{a}$ varies as an n-tuple in M according to the respective quantifiers (e.g. if $Q_i \equiv \forall$, then a_i ranges across M, and similarly for when Q_i is $\exists$), and in particular, each $\bar{a}$-substitution corresponds to a map $\sigma : \mathbb{N} \to M$, via $v_i \mapsto a_i$. Thus, whenever $\mathcal{M}$ satisfies a sentence with universal quantifiers, there must be a corresponding family of substitution maps $\sigma : \mathbb{N} \to M$ ranging through all of M on the i^{th} component whenever Q_i is a universal quantifier.

Since each substitution can be identified with a branch in $M^{\mathbb{N}}$, whenever M is a set, the family of substitution maps for each sentence ϕ can be understood as a set.

When we pass from some underlying set-theoretic universe to a Class-sized universe, we must address the issue of the scope of our quantifiers and definability of truth with respect to sets and classes. As described above, classical model theory is unable to address this, as models are sets, satisfying sets of formula, written in signatures composed of sets of symbols. Morse-Kelley set theory with Global Choice is one candidate for a set-theory where truth is generally definable for class-structures and assertions about global truth can freely apply induction within class-structures. However, Ehrlich demonstrated in [13] that NBG, a conservative extension of ZFC, is sufficient for defining truth with respect to No, since enough basic algebra and model theory can be developed within No to handle truth for definable structures with Class-sized universes relative to the surreal numbers, such as real-closed exponential fields [2] or homogeneous H-fields [6].

Given a relational language $\mathcal{L}$ of order type $\mu \in \mathrm{On}$, where each R_α has finite arity on A of power $\leq \mathrm{On}$, an $\mathcal{L}$-structure $\mathfrak{A}$ inside NBG is the following class:

$$(A \times \{0\}) \cup \mathfrak{R}$$

where

$$\mathfrak{R} := \bigcup \{R_\alpha \times \{\alpha\} \colon \alpha \in \mu\}.$$

This will suffice to handle cases where our underlying universe is a Class-sized object, i.e. $|A| = \mathrm{On}$. Further, given our interest in inductive theories $T \supset \mathrm{RCF}$, for $\mathcal{L}$-structures $\mathfrak{A}$ such that $\mathfrak{A}_\beta \subset \mathfrak{A}$ are substructures for each $\beta \in \mathrm{On}$, and $\bigcup_{\mathrm{On}} \mathfrak{A}_\beta = \mathfrak{A}$, we can define truth in Class-sized universes as follows:

Let ϕ, ψ denote quantifier free formula.

1. If $\varphi = \phi$ or $\varphi = \exists \bar{x} \psi$, $\mathfrak{A} \models \varphi$ if and only if φ is true in some $\mathfrak{A}_\beta$, i.e. $\mathfrak{A}_\beta \models \varphi$ as in classical model theory.
2. If $\varphi = \forall \bar{x} \exists \bar{y} \psi$, then $\mathfrak{A} \models \varphi A$ if and only if for each $\bar{a} \in A$, $\bar{a} \in \mathfrak{A}_\beta$ for some β, and $\mathfrak{A}_\beta \models \exists \bar{y} \psi(\bar{a})$.

Because we are able to identify Class-objects up to isomorphism in NBG, when working with relational languages $\mathcal{L}$ that are proper sets, we can extend the classical notions of homogeneity and saturation from the cases where κ is some proper cardinal, to cases where κ can be identified with On. Ehrlich names these corresponding notions *absolutely homogeneous universal models* (respectively *absolutely saturated universal models*) $\mathcal{M}$ satisfying a theory T. Explicitly,

Definition 50. *Suppose* T *is an* $\mathcal{L}$*-theory and* $\mathcal{M} \models \mathsf{T}$ *with universe* M*. Suppose* $\kappa \leq \mathrm{On}$*.*

We say $\mathcal{M}$ *is* κ***-universal*** *with respect to* T *if the models of* T *are such that every* $\mathcal{N}$ *such that* $|\mathsf{N}| < \kappa$*, then* $\mathcal{N} \prec \mathcal{M}$*.*

$\mathcal{M}$ *is* κ***-homogeneous*** *if for every* $\mathsf{A} \subset \mathsf{M}$ *such that* $|\mathsf{A}| < \kappa$*, and* $\mathsf{f} : \mathsf{A} \to \mathsf{M}$ *is a partial elementary map, and* $\mathsf{a} \in \mathsf{M}$*, there is* $\mathsf{f}^* \supseteq \mathsf{f}$ *such that* $\mathsf{f}^* : \mathsf{A} \cup \{\mathsf{a}\} \to \mathsf{M}$ *will also be partial elementary, i.e. every isomorphism between substructures of* $\mathcal{M}$ *can be extended to an automorphism.*

$\mathcal{M}$ *is a* κ***-saturated*** *if for all* $\mathsf{A} \subset \mathsf{M}$ *such that* $|\mathsf{A}| < \kappa$*, and all* $\mathsf{p} \in \mathsf{S}_n^{\mathcal{M}}$*, then* p *is realized in* $\mathcal{M}$*.*

Whenever $|\mathsf{M}| = \kappa$*, we drop the* κ*, i.e.* $\mathcal{M}$ *is simply, universal/homogeneous/saturated.*

Further, for cardinals $\kappa \in \mathrm{On}$,

Theorem 38. *For* $\kappa \geq \aleph_0$*, the following equivalent:*

1. $\mathcal{M}$ *is* κ*-saturated.*
2. $\mathcal{M}$ *is* κ*-homogeneous and* κ^+*-universal.*

If $\kappa \geq \aleph_1$*, then (1) and (2) are equivalent to*

3. $\mathcal{M}$ *is* κ*-homogeneous and* κ*-universal.*

These concepts can be extended to the case of a Class-sized model; the following theorem summarizing Theorems 1 and 2 of [13] is instrumental for the overarching argument of this dissertation:

Theorem 39. *If* $|\mathcal{L}| < \text{On}$, *then*

1. *If* T *is Jónsson in* $\mathcal{L}$, *then there is, up to isomorphism, an unique model that is absolutely homogeneous universal with respect to* T*;*
2. *If* T *is a complete, model-complete theory in* $\mathcal{L}$ *with an infinite model, then up to isomorphism, there is an unique absolutely saturated model of* T.

CHAPTER 3

PROPERTIES OF GENETIC FUNCTIONS

3.1 Genetic Functions

Definition 51. *We say that a representation of games of the form* $\{\mathsf{L}\}|\{\mathsf{R}\}$ *is given by* ***genetic formula****, with the phrasing inspired by the fact that each game has a* birthday*. The formulae* L *and* R *define the Left and Right options for a game, and modulo the notion of equality above, the game defined by* L *and* R *are the unique such game. Below are several genetic formulae defining* ***compounds*** *of games, with description and notation borrowed from [5] or otherwise from [23]:*

1. ***Disjunctive*** *move in exactly one component,*

$$\mathsf{G}+\mathsf{H}\equiv\left\{\mathsf{G}^{\mathsf{L}}+\mathsf{H},\mathsf{G}+\mathsf{H}^{\mathsf{L}}\right\}|\left\{\mathsf{G}^{\mathsf{R}}+\mathsf{H},\mathsf{G}+\mathsf{H}^{\mathsf{R}}\right\}$$

2. ***Conjunctive*** *Move in all components at once, where play ends whenever one component terminates*

$$\mathsf{G}\wedge\mathsf{H}\equiv\left\{\mathsf{G}^{\mathsf{L}}\wedge\mathsf{H}^{\mathsf{L}}\right\}|\left\{\mathsf{G}^{\mathsf{R}}\wedge\mathsf{H}^{\mathsf{R}}\right\}$$

3. ***Selective*** *Move in any number of components, but at least one component*

$$\mathsf{G}\vee\mathsf{H}\equiv\left\{\mathsf{G}^{\mathsf{L}}\vee\mathsf{H},\mathsf{G}\vee\mathsf{H}^{\mathsf{L}},\mathsf{G}^{\mathsf{L}}\vee\mathsf{H}^{\mathsf{L}}\right\}|\left\{\mathsf{G}^{\mathsf{R}}\vee\mathsf{H},\mathsf{G}\vee\mathsf{H}^{\mathsf{R}},\mathsf{G}^{\mathsf{R}}\vee\mathsf{H}^{\mathsf{R}}\right\}$$

4. ***Diminished disjunctive*** *Move in exactly one component with play ending immediately whenever any one component terminates*

$$G \boxplus H \equiv \begin{cases} 0 & G \equiv 0 \text{ or } H \equiv 0 \\ \left\{G^L \boxplus H, G \boxplus H^L\right\} \mid \left\{G^R \boxplus H, G \boxplus H^R\right\} & o.w. \end{cases}$$

5. ***Continued conjunctive*** *Move in all non-terminal components with play ending only when all components terminate*

$$G \triangledown H \equiv \begin{cases} G + H & G \equiv 0 \text{ or } H \equiv 0 \\ \left\{G^L \triangledown H^L\right\} \mid \left\{G^R \triangledown H^R\right\} & \end{cases}$$

6. ***Shortened selective*** *Move in any number of components, with play ending immediately when any one of them terminates.*

$$G \vartriangle H \equiv \begin{cases} 0 & G \equiv 0 \text{ or } H \equiv 0 \\ \left\{G^L \vartriangle H, H \vartriangle G^L, H^L \vartriangle G^L\right\} \mid \left\{G^R \vartriangle H, G \vartriangle H^R, G^R \vartriangle H^R\right\} & o.w. \end{cases}$$

7. ***Ordinal*** *Move in* G *or* H*, but any move on* G *annihilates* H

$$G : H \equiv \left\{G^L, G : H^L\right\} \mid \left\{G^R, G : H^R\right\}$$

8. ***Side*** *Move in* G *or* H*, with Left's moves on* H *annihilating* G*, and Right's moves on* G *annihilating* H

$$\mathsf{G} \diamond \mathsf{H} \equiv \left\{\mathsf{G}^{\mathsf{L}} \diamond \mathsf{H}, \mathsf{H}^{\mathsf{L}}\right\} \mid \left\{\mathsf{G}^{\mathsf{R}}, \mathsf{G} \diamond \mathsf{H}^{\mathsf{R}}\right\}$$

9. ***Sequential*** *Move in* G *unless* G *has terminated, in which case move in H.*

$$\mathsf{G} \rightarrow \mathsf{H} \equiv \begin{cases} \mathsf{H} & \mathsf{G} \equiv 0 \\ \left\{\mathsf{G}^{\mathsf{L}} \rightarrow \mathsf{H}\right\} \mid \left\{\mathsf{G}^{\mathsf{R}} \rightarrow \mathsf{H}\right\} & \text{o.}w. \end{cases}$$

We can also construct more sophisticated compounds, such as the ***Conway product****,*

$$\mathsf{GH} \equiv \left\{\mathsf{G}^{\mathsf{L}}\mathsf{H} + \mathsf{G}\mathsf{H}^{\mathsf{L}} - \mathsf{G}^{\mathsf{L}}\mathsf{H}^{\mathsf{L}}, \mathsf{G}^{\mathsf{R}}\mathsf{H} + \mathsf{G}\mathsf{H}^{\mathsf{R}} - \mathsf{G}^{\mathsf{R}}\mathsf{H}^{\mathsf{R}}\}\right|$$

$$\{\mathsf{G}^{\mathsf{L}}\mathsf{H} + \mathsf{G}\mathsf{H}^{\mathsf{R}} - \mathsf{G}^{\mathsf{L}}\mathsf{H}^{\mathsf{R}}, \mathsf{G}\mathsf{H}^{\mathsf{L}} + \mathsf{G}^{\mathsf{R}}\mathsf{H} - \mathsf{G}^{\mathsf{R}}\mathsf{H}^{\mathsf{L}}\}.$$

Remark 18. *One principle motivation to study when genetic functions are defined, when they become fuzzy, and when they become undefined, stems from the ubiquity of genetic functions whose term construction depends on the Conway product. Notably, the Conway product fails to satisfy the uniformity property when taken over arbitrary partizan games, while it does satisfy the uniformity property when restricted to the Class of surreal numbers. This is because there are partizan games* $\mathsf{G}_1, \mathsf{G}_2$*, and* H *such that* $\mathsf{G}_1 = \mathsf{G}_2$ *but* $\mathsf{G}_1\mathsf{H} \neq \mathsf{G}_2\mathsf{H}$ *under the definition of* $=$

modulo disjunctive sums. This occurs since we only need $G_1 \geq G_2$ *and* $G_2 \geq G_1$*,*[1] *which does not preclude the possibility that the games can be confused.*

We eliminate the possibility that any two games can be confused by restricting to the subclass of surreal numbers (by restricting ourselves to a linearly ordered subclass of partizan games): if for two numbers G_1, G_2 *we have* $G_1 \geq G_2$ *and* $G_2 \geq G_1$*, then* $G_1 = G_2$ *as numbers. Furthermore, the Conway product satisfies the ordinary axioms for commutative ring multiplication with respect to the defined equivalence relation = used in the literature of combinatorial game theory (that is, if* x, y, z *are surreal numbers such that* $x = y$*, then* $xz = yz$*), and satisfies the linear order relation. Specifically, because each number* $a \in \mathrm{No}$ *is the unique such number of minimal length that* $L_a < a < R_a$*, and for all* $a, b, c, d \in \mathrm{No}$ *such that* $a > b$ *and* $c > d$*, we have* $ac - bc > ad - bd$*, from which we may prove the uniformity property for multiplication when restricting to the surreal numbers (see Theorem 3.4 of [1] for details). Thus, we have that the Conway product is a genetic function from* $\mathrm{No} \to \mathrm{No}$*, but this will not extend to a genetic function in* $\mathrm{PG} \to \mathrm{PG}$*.*

For this reason, we first will provide a definition for genetic functions for the Class of partizan games, before examining the definition of genetic functions given by [3]. The principle restriction introduced by [3] is the order property. We will show that the order property implies the uniformity property necessary to be a genetic function. Furthermore, the Class of genetic functions they define will have their codomain contained in the surreal numbers.

---

[1] *Recall that* $G_1 \leq G_2$ *is equivalent to saying there is no inequality of the form* $G_1^R \leq G_2$ *or* $G_1 \leq G_2^L$*.*

The construction given in [3] is sufficient to investigate the model theoretic properties of the surreal numbers because the Class of functions will be defined for arbitrary first-order function symbols that are entire, from which we may safely restrict to initial subtrees of the Class of surreal numbers that will be closed under said function symbols.

In order to prove that the standard field operations and various genetic functions that arise in constructing the surreal numbers (such as the ω function, or exp) are well defined, and possess the properties we claim they do, it is necessary to clarify the notions of cofinal and coinitial representative sets of Left and Right options respectively.

Definition 52. *Suppose* $x \in \mathrm{PG}$ *has a canonical presentation* $x = \{x^L\} \mid \{x^R\} = L_x|R_x$*, and let subset* $F \subsetneq \mathrm{PG}$*. We say* F *is cofinal in* L_x *if for every* $y \in L_x$*, there is* $f \in F$ *such that* $y \leq f$*. Similarly,* F *is coinitial in* R_x *if for every* $y \in R_x$ *there is* $f \in G$ *such that* $f \leq y$*.*

In particular, every surreal number $a \in \mathrm{No}$ *has a canonical representation where* $L_a = \{x \in \mathrm{No} \colon x <_s a \wedge x < a\}$ *and* $R_a = \{x \in \mathrm{No} \colon x <_s a \wedge a < x\}$*.*

We condense the cofinality theorems of Gonshor [1] (see Theorems 2.6-2.9) in the following result:

Theorem 40. *Suppose* $a \in \mathrm{No}$ *with sets* F_1, F_2, G_1, G_2 *of surreal numbers.*

1. *Suppose* $a = F_1|G_2$*. If* $F_2 < a < G_2$*, and* (F_2, G_2) *is cofinal in*[1] (F_1, G_1)*, then* $a = F_2|G_2$*.*

2. *Suppose that* (F_1, G_1) *and* (F_2, G_2) *are mutually cofinal. Then* $F_1|G_1 = F_2|G_2$*.*

---

[1] *Any ordered pair of sets* (F_2, G_2) *of surreal numbers will be cofinal in another pair* (F_1, G_1) *provided* F_2 *is cofinal in* F_1 *and* G_2 *is coinitial in* G_1*.*

3. *Suppose* $F_1 = \{b \in \mathrm{No} \colon b < a \wedge b <_s a\}$ *and* $G_1 = \{b \in \mathrm{No} \colon b > a \wedge b <_s a\}$. *Then* $a = F_1|G_1$. *(In fact, these are the canonical representations of* a*, and without loss of generality, we denote these by* L_a, R_a*.)*

4. *(Inverse Cofinality) Let* $a = L_a|R_a = F|G$. *Then* (F, G) *is cofinal in* (L_a, R_a).

We will need to generalize these results for our analysis of arbitrary genetic functions.

Definition 53. *For any* $\bar{a} \in \mathrm{No}^n$*, the canonical Left and Right sets are*

$$L_{\bar{a}} := \{\bar{x} \in \mathrm{No}^n \colon (\forall i \leq n(x_i \leq_s a_i \wedge x_i \leq a_i)) \wedge \exists i \leq n(x_i <_s a_i \wedge x_i < a_i)\}$$

and

$$R_{\bar{a}} := \{\bar{x} \in \mathrm{No}^n \colon (\forall i \leq n(x_i \leq_s a_i \wedge a_i \leq x_i)) \wedge \exists i \leq n(x_i <_s a_i \wedge a_i < x_i)\}.$$

A set $F \subset \mathrm{No}^n$ *is cofinal in* $L_{\bar{a}}$ *if for every* $\bar{a}' \in L_{\bar{a}}$*, there is some* $\bar{f} \in F$ *such that componentwise for every* $i \leq n$*,* $a'_i \leq f_i$*. Similarly, we say* G *is coinitial in* $R_{\bar{a}}$ *if for every* $\bar{a}'_i \in R_{\bar{a}}$*, there is* $\bar{g} \in G$ *such that componentwise* $g_i \leq \bar{a}_i$*.*

Proposition 9. *The Cartesian product is not a game compound, in the sense that as a Class* $\mathrm{PG} \times \mathrm{PG} \not\subset \mathrm{PG}$*, i.e. these are distinct Classes with the Class-sized Cartesian product is not a proper subClass of* PG *nor is it equivalent to* PG*.*

Proof. First, despite having superficially similar sets of Left and Right options, because the standard ordering of Partizan games also applies to the Selective compound, it is not directly comparable to the Cartesian product. For example, $1 \vee -1 \| 0$, with the ordering on Partizan games. If the Left player moves first, then he chooses 1; if the Right player moves first, she chooses -1. Thus, we cannot say that option sets are cofinal nor coinitial.

Moreover, the Cartesian product is not an operator sending Partizan games to Partizan games. If it were, for example, then the ordering on Partizan games entails that $(0, 0) = 0$. This is not an issue with Classes, since we can restrict our attention from the Class of Partizan games, and just focus on the $\bigcup_{n \in \omega} \mathrm{PG}_n$, which is a proper set.

□

Because of Proposition 9, we will need to extend the ordering on Partizan games to a lexicographical ordering on the components of No^n. With this ordering, we can then extend the cofinality and coinitiality results from [1] as follows:

Theorem 41. *1. Supposing for $\bar{a} \in \mathrm{No}^n$ we have $a_i = F_i | G_i$ for each $i \leq n$, and $F'_i < a_i < G'_i$ with (F'_i, G'_i) cofinal in (F_i, G_i) for each $i \leq n$. Then each $a_i = F'_i | G'_i$, and moreover, $\bar{a} = \prod F_i | \prod G_i = \prod F'_i | \prod G'_i$. Furthermore, supposing $F' \subset \mathrm{No}^n$ and $G' \subset \mathrm{No}^n$, such that the projected components for each i are as above, then $\bar{a} = F' | G'$.*

2. If for $i \leq n$, we have (F_i, G_i) and (F'_i, G'_i) are mutually cofinal/coinitial in one another, then $F_i | G_i = F'_i | G'_i$ for each $i \leq n$, and thus $\prod F_i | \prod G_i = \prod F'_i | \prod G'_i$.

3. Let $\bar{a} = L_{\bar{a}} | R_{\bar{a}} = F | G$. Then (F, G) is cofinal in $(L_{\bar{a}}, R_{\bar{a}})$.

Proof. Each of these bullet points follows directly from Theorem 40 applied componentwise. However, for completeness, a brief summary is in order:

1. Supposing that there is some $\bar{b} \in \mathrm{No}^n$ such that $\iota(\bar{b}) < \iota(\bar{a})$ and $\prod F'_i < \bar{b} < \prod G'_i$ with the natural lexicographical ordering. Then by cofinality, $\prod F'_i < \bar{a} < \prod G'_i$, which will contradict the minimality of $\iota(\bar{a})$, as for at least one component, $\iota(b_i) < \iota(a_i)$. Hence, $\bar{a} = \prod F'_i | \prod G'_i$. The "furthermore" follows immediately.

2. By mutual cofinality, we have that $\{x_i \colon F_i < x_i < G_i\} = \{x_i \colon F'_i < x_i < G'_i\}$, and componentwise, each will have the same minimal element, whence $F|G = F'|G'$.

3. Suppose that $\bar{b} \in L_{\bar{a}}$ and $\bar{b} < \bar{a} < G$. Since $\bar{a}$ is of minimal length among elements such that each component of $\bar{a}$ is of minimal length satisfying $F_i < a_i < G_i$ for each $i \leq n$, and each $\bar{b}$ is a predecessor of $\bar{a}$, $F' < \bar{b}$ is impossible. Precisely, since $\iota(b) < \iota(a)$, by minimiality, there must exist $c \in F_i$ such that $c \geq b_i$.

   We apply a similar argument for $R_{\bar{a}}$ and G, concluding that (F, G) is cofinal in $(L_{\bar{a}}, R_{\bar{a}})$.

□

Definition 54. *Suppose* $f : \mathrm{PG} \to \mathrm{PG}$ *is a recursively defined function in terms of the set of Left and Right options of a game* a*. We denote this by saying that* f^L *is a* ***Left option*** *drawn from the set* $L_f(u, v)$*, where* u *and* v *indicate indeterminates for the Left and Right options defining an argument. We similarly define* ***Right options*** f^R*. We let* $L_f(L_a, R_a)$ *denote the union of the sets of Left options of* $f(a)$ *as the terms with Left options from* L_a *and Right options from* R_a *vary. Similarly for* $R_f(L_a, R_a)$*, as well as for other sets* G, H *that are cofinal and coinitial in* L_a *and* R_a *respectively.*

We say that such recursively defined functions have the ***uniformity property*** *if for all* $a \in Dom f$*, and whenever* $a = L_a | R_a = G|H$ *such that* G *is cofinal in* L_a *and* H *is coinitial in* R_a*, then* $L_f(G, H)$ *is cofinal in* $L_f(L_a, R_a)$ *and* $R_f(G, H)$ *is coinitial in* $R_f(L_a, R_a)$ *so that the target game value defined is invariant under the representation of the source game value. Whenever* f *is a recursively defined function with the uniformity property, we say that* g *is a* ***genetic function****.*

Remark 19. *The uniformity property is what allows us to consider these Class functions as recursively well-defined functions, and also allows for the Class of genetic functions to be closed under composition.*

Proposition 10. *The following functions are genetic functions:*

1. *The zero-function is genetic;*
2. *The identity function,* $\text{ID}(x) := \left\{x^L\right\} | \left\{x^R\right\}$*;*
3. *The successor function* $S(x) := \{x\} | \left\{x^R\right\}$*;*
4. *Projections are genetic functions.*

Proof. 1. Let $\mathbf{0}(x) := \{\}|\{\} = 0$ for all games x. This is allowed as we may choose to define the function without selecting any options of x. This satisfies the uniformity property vacuously, as the result will be defined for all mutually cofinal and coinitial sets of options defining a game.

2. This follows immediately by construction. If $x = \left\{x^L\right\} \mid \left\{x^R\right\} = L_x|R_x = F|G = \{f\} \mid \{g\}$, then by the cofinality and inverse cofinality theorems of [1] we witness the uniformity property, as
$$\text{ID}(x) = \text{ID}(L_x|R_x) = \left\{x^L\right\} \mid \left\{x^R\right\} = \{f\} \mid \{g\} = \text{ID}(F|G).$$

3. For every $x = \left\{x^L\right\} \mid \left\{x^R\right\} = \{f\} \mid \{g\}$, we have by the cofinality and inverse cofinality theorems of [1] that
$$S(x) = \{x\} \mid \left\{x^R\right\} = \left\{(\left\{x^L\right\} \mid \left\{x^R\right\})\right\} \mid \left\{x^R\right\} = \{(\{f\} \mid \{g\})\} \mid \{g\} = S(x).$$
Further inspection shows that when restricting x to the ordinals, that this is precisely the standard successor map, while extending S to the surreal numbers in general produces the **right successor** branch of the standard s-hierarchy, as we're choosing the simplest element to the right of the node x that is less than all right options of x^R.

4. Given $\bar{x} = L_{x_1} \times L_{x_2} \times \cdots \times L_{x_n} \mid R_{x_1} \times \cdots \times R_{x_n}$, let $\pi_i(\bar{x}) = L_{x_i}|R_{x_i}$. This will have the uniformity property by item (2).

□

Remark 20. *One can use ordinal and sequential compound forms of games and the genetic functions listed in Proposition 10 to recover all primitive recursive functions. Including the details of this argument go beyond the scope of this dissertation. If the reader is uncomfortable with this Remark, then please regard this as a conjecture.*

We now provide a recursive definition of the Class of genetic functions defined with respect to the surreal numbers proper:

Definition 55. *Let* $\mathfrak{G}$ *be a Class of genetic functions defined on all of* No *such that if* $f \in \mathfrak{G}$*, then* $f(\mathrm{No}) \subset \mathrm{No}$*. We define a new genetic function* $f : \mathrm{No} \to \mathrm{No}$ *in one of two ways:*

<u>*Composition*</u> *If* $g, h \in \mathfrak{G}$*, set* $f = g \circ h$*. If for all* $x \in \mathrm{No}$*,* $g(x) := L_g(x^L, x^R)|R_g(x^L, x^R)$ *and similarly,* $h(x)$ *is given in terms of sets of genetic Left and Right formula given with respect to the Left and Right options of* x*, then*

$$f(x) := \left\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \bigcup_{\substack{h^L \in L_h(x^L, x^R) \\ h^R \in R_h(x^L, x^R)}} \{g^L(h(x)) \colon g^L \in L_g(h^L(x), h^R(x))\} \right\} \Bigg|$$

$$\left\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \bigcup_{\substack{h^L \in L_h(x^L, x^R) \\ h^R \in R_h(x^L, x^R)}} \{g^R(h(x)) \colon g^L \in L_g(h^L(x), h^R(x))\} \right\}.$$

<u>*Adjunction*</u> *Let* v, w *denote indeterminates, and let* $f : \mathrm{No} \to \mathrm{No}$ *be a function symbol. We outline this process in the following stages:*

1. *We form the Ring* $K := \mathrm{No}[\{g(v), g(w) \mid g \in S \cup \{f\}\}]$*, where* S *is a set closed under composition consisting of previously defined genetic functions on one variable.*

2. *We obtain a Class*

$$S(v,w) = \{c_1 + c_2 h(c_3 x + c_4) \colon c_1, c_2, c_3, c_4 \in K, h \in S\}.$$

3. *We then form Ring* $R(v,w) := \mathrm{No}[S(v,w)]_{P_S}$*, where* P_S *is the cone of strictly positive polynomials with function from* S*.*

4. *We then choose proper subsets* $L_f, R_f \subset R(v,w)$*.*

5. *Fix an* $x \in \mathrm{No}$*, and supposing*[1] $f(y)$ *has already been defined for all* $y \in L_x \cup R_x$*, we substitute* v *with* x^L *and* w *with* x^R *in* $R(v,w)$*, obtaining the set of functions* $L_f(x^L, x^R), R_f(x^L, x^R) : \mathrm{No} \to \mathrm{No}$*.*

6. *Next, provided that the* ***order condition*** *holds, i.e. for all* $x^L, x^{L'} \in L_x$ *and* $x^R, x^{R'} \in R_x$*, and* $f^L \in L_f(x^L, x^R)$ *and* $f^R \in R_f(x^{L'}, x^{R'})$ *we have* $f^L(x) < f^R(x)$*,*

$$f(x) := \left\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^L(x) \colon f^L \in L_f(x^L, x^R)\} \right\} \Big| \left\{ \bigcup_{\substack{x^L \in L_x \\ x^R \in R_x}} \{f^R(x) \colon f^R \in R_f(x^L, x^R)\} \right\}$$

7. *Finally, let the global* ***cofinality condition*** *be the following:*

$$\forall x, y, z \in \mathrm{No}((y < x < z) \to L_f(y,z)[x] < f(x) < R_f(y,z)[x],$$

---

[1] *Given any pair of subsets* (L_f, R_f) *of* $R(v,w)$*, we can always define* $f(0)$ *by considering the cut of the terms that can be evaluated, i.e. whatever terms are strictly constant.*

i.e. for all $y < x < z$*, the set of values of* $f^L(y, z; x) \in L_f(y, z)$ *(similarly for the Right option set), is such that every* $f^L(y, z; x) < f(x)$ *(similarly every Right option in the Right option set is evaluated at* x *to be greater than* $f(x)$*). Once* f *is globally defined over* No*, we prove that the cofinality condition holds, typically via induction with respect to the natural sum of the lengths of the arguments, as is done in the case of addition and multiplication.*

Then $f \in \mathfrak{G}$*.*

We similarly define genetic functions of higher arity via composition and adjunction, to define the Class of higher genetic functions $\mathcal{G}$*.*

<u>*Composition*</u> *If* $g, h \in \mathfrak{G}$*, such that* g *has arity n and each* h *has arity* m*, we can set* $f = g(h_1, \ldots, h_n)$*. If for all* $\bar{x} \in \mathrm{No}^n$*,* $g(\bar{x}) := L_g(\bar{x}^L, \bar{x}^R) | R_g(\bar{x}^L, \bar{x}^R)$ *and similarly, each* $h_i(\bar{x})$ *is given in terms of sets of genetic Left and Right formula given with respect to the Left and Right options of* $\bar{x}$*, then*

$$f(\bar{x}) := \left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \bigcup_{i=1}^{n} \bigcup_{\substack{h_i^L \in L_{h_i}(\bar{x}^L, \bar{x}^R) \\ h_i^R \in R_{h_i}(\bar{x}^L, \bar{x}^R)}} \{g^L(h_1(\bar{x}), \ldots, h_n(\bar{x})) \colon g^L \in L_g((h_1^L(\bar{x}), \ldots, h_n^L(\bar{x})), (h_1^R(\bar{x}), \ldots, h_n^R(\bar{x})))\} \right\} \Bigg|$$

$$\left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \bigcup_{i=1}^{n} \bigcup_{\substack{h_i^L \in L_{h_i}(\bar{x}^L, \bar{x}^R) \\ h_i^R \in R_{h_i}(\bar{x}^L, \bar{x}^R)}} \{g^R(h_1(\bar{x}), \ldots, h_n(\bar{x})) \colon g^R \in R_g((h_1^L(\bar{x}), \ldots, h_n^L(\bar{x})), (h_1^R(\bar{x}), \ldots, h_n^R(\bar{x})))\} \right\}.$$

<u>*Adjunction*</u> *Let* $\bar{v}, \bar{w}$ *denote tuples of indeterminates, and let* $f : \mathrm{No}^n \to \mathrm{No}$ *be a function symbol of arity* n*. We outline this process in the following stages:*

1. *We form the Ring*

$$K := \mathrm{No}[\{g(\bar{u}) \mid g \in \mathfrak{G} \cup \{f\}, \bar{u} = \langle u_1, \dots, u_{n_g} \rangle, u_i \in \{v_i, w_i\}\}].$$

2. *We obtain a Class*

$$\mathcal{G}(\bar{v}, \bar{w}) = \{c_1 + c_2 h(A\bar{x} + c_3) \colon h \in \mathcal{G}, i \in [n_h], c_1, c_2 \in K, c_3 \in K^{n_h}, A \in K^{n_h \times |\bar{x}|}\}.$$

3. *We then form Ring* $R(\bar{v}, \bar{w}) := \mathrm{No}[\mathfrak{G}(\bar{v}, \bar{w})]_{P_S}$.

4. *We then choose proper subsets* $L_f, R_f \subset R(\bar{v}, \bar{w})$.

5. *Fix an* $\bar{x} \in \mathrm{No}^{n_f}$*, and supposing* $f(\bar{y})$ *has already been defined for all* $\bar{y} \in L_{\bar{x}} \cup R_{\bar{x}}$*, we substitute instances of* $\bar{v}$ *with* $\bar{x}^L$ *and* $\bar{w}$ *with* $\bar{x}^R$ *in* $R(v, w)$*, obtaining a resulting set of functions* $L_f(\bar{x}^L, \bar{x}^R), R_f(\bar{x}^L, \bar{x}^R)$ *by substitution. Importantly, in the multivariable setting, we may have substitutions entirely from a Left set, a Right set, or a combination of both. In the cases that the options are exclusively drawn from the Left set (or the Right set),* $\bar{x}^L$ *is a tuple where at least one coordinate is a strict Left predecessor (or Right predecessor). In the case where we have mixed options, at least one substitution must be a proper predecessor drawn from* $L_{x_i} \cup R_{x_i} = \mathrm{pr}(x_i)$*, drawing from* L_{x_i} *if* v_i*, and* R_{x_i} *otherwise if substituting* w_i*.*

6. *Next, provided that the* ***order condition*** *holds, i.e. for all* $\bar{x}^L, \bar{x}^{L'} \in L_{\bar{x}}$ *and* $\bar{x}^R, x^{R'} \in R_{\bar{x}}$*, and* $f^L \in L_f(\bar{x}^L, \bar{x}^R)$ *and* $f^R \in R_f(\bar{x}^{L'}, \bar{x}^{R'})$ *we have* $f^L(\bar{x}) < f^R(\bar{x})$*,*

$$f(\bar{x}) := \left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \{f^L(\bar{x}) \colon f^L \in L_f(\bar{x}^L, \bar{x}^R)\} \right\} \mid \left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \{f^R(\bar{x}) \colon f^L \in R_f(\bar{x}^L, \bar{x}^R)\} \right\}.$$

To simplify this notation going forward, we let

$$L_f(L_{\bar{x}}, R_{\bar{x}}; \bar{x}) := \left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \{f^L(\bar{x}) \colon f^L \in L_f(\bar{x}^L, \bar{x}^R)\} \right\}$$

and

$$R_f(L_{\bar{x}}, R_{\bar{x}}; \bar{x}) := \left\{ \bigcup_{\substack{\bar{x}^L \in L_{\bar{x}} \\ \bar{x}^R \in R_{\bar{x}}}} \{f^R(\bar{x}) \colon f^L \in R_f(\bar{x}^L, \bar{x}^R)\} \right\}.$$

7. *Finally, we prove that the* ***cofinality condition*** *holds, i.e.*

$$\forall \bar{x}, \bar{y}, \bar{z} \in \mathrm{No}^{|f|}((\bigwedge_{i \in [|f|]} (y_i \leq x_i \leq y_i) \wedge \bigvee_{i,j,k \in [|f|]} (y_j < x_i < z_k)) \to$$

$$(L_f(\bar{y}, \bar{z})[\bar{x}] < f(\bar{x}) < R_f(\bar{y}, \bar{z})[x]))$$

Then $f \in \mathfrak{G}$*.*

Finally, without loss of generality, we may omit $(\bar{-})$ *when describing arbitrary genetic functions, unless necessary.*

Remark 21. *We have added the cofinality condition to the construction of the Class of genetic functions from [3] because their definition is inadequate to guarantee uniformity. It evidently does not by Theorem 3, as the* ι *function defined there vacuously satisfies the order condition, but not the cofinality condition.*

Following [4], we impose the so-called global cofinality condition to ensure uniformity. The intuition behind this condition is that we wish for all $y \in \mathfrak{L}_x$ *and all* $z \in \mathfrak{R}_x$ *the corresponding option sets*

$$\left(\bigcup_{\mathfrak{L}_x, \mathfrak{R}_x} \mathsf{L}_f(y, z)[x]\right) \subset \mathfrak{L}_{f(x)}$$

and similarly for the Right options.

The following theorem establishes that for surreal-valued recursively defined functions over the surreal numbers, the order condition and cofinality condition is equivalent to a recursively defined surreal-value function with the uniformity property.

Theorem 42. *Suppose* f *is defined with respect to option term sets* $\mathsf{L}_f, \mathsf{R}_f \subset \mathsf{R}(v, w)$ *as above. Then* f *is a surreal-valued genetic function if and only if* f *has the uniformity property.*

Proof. In the forward direction, suppose that f is a surreal-valued genetic function. Then f satisfies the order condition and the cofinality condition. We wish to show that for all $a \in \mathbf{No}$ and all representations $(F|G)$ of a in $\mathfrak{E}^*$, we have

$$\begin{aligned} f(a) &= \left\{ \bigcup_{\substack{b \in F \\ c \in G}} \{f^L(x) \colon f^L \in L_f(b,c)\} \right\} \Big| \left\{ \bigcup_{\substack{b \in F \\ c \in G}} \{f^R(x) \colon f^R \in R_f(b,c)\} \right\} \\ &= L_f(L_a, R_a; a) | R_f(L_a, R_a; a) \end{aligned}$$

But this follows by the global cofinality condition.

In the reverse direction, if f is a recursively defined surreal-valued function invariant under representation defined by $L_f, R_f \subset R(v, w)$, then necessarily the order property must be satisfied, as otherwise there would be some Left option and Right option and $b < a < c$ such that $f^L(b, c; a) \geq f^R(b, c; a)$, whence the game value defined will not be numeric, or $f^L(b, c; a) \geq f(a)$ or $f^R(b, c; a) \leq f(a)$.

□

Remark 22. *While the construction above concerns a ring of genetic functions closed under composition, if we relax our requirement to study entire functions, we can provide definitions of partial genetic functions, such as* $\frac{1}{x}$ *or* $\log(x)$ *whose images lie in the surreal numbers by way of genetic formula and by stipulating that we are to ignore options, such as when division by* *zero occurs. For example, we can define* y *to be the multiplicative inverse of* x *whenever* $x > 0$ *(and similarly, whenever* $x < 0$*) by*

$$y = \left\{ 0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \right\} \mid \left\{ \frac{(1 + x^L)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R} \right\}.$$

This construction requires that we have x^L *range over all positive elements of* x*, and with the reciprocals of* x^L, x^R *recursively defined, along with the* y^L, y^R*, the definition of* y *is recursive with the closure formed by iteratively feeding the options constructed at each stage back into the definition. This construction, from [5] and [23], differs only slightly from the construction in [1], where inverses are defined with respect to successive approximations of the left and right options of* y*.*

It is important to note that if we are to define these options in the Class of partizan games, the multiplicative inverse is still not defined at 0. It is of great importance to find conditions for genetic formulae under which a partial function maximally defined with respect to the surreals so that its range lies in the surreals, can be extended to an entire function on the surreals such that complements of the domain of the function are mapped to some non-surreal partizan game.

A few immediate consequences to this definition of entire genetic functions provided by [3] are summarized below:

Lemma 7. *1. For* $\mathfrak{S} \subset \mathcal{G}$*, the Class of single variable genetic functions is closed under addition, multiplication and composition, so in particular all polynomial functions with surreal coefficients are genetic.*

2. *Given explicit restrictions on domain, rational polynomial functions are also genetic given the genetic definition of* $f(x) = \frac{1}{x}$.

3. *Power series of limit ordinal index are also genetic functions.*

4. *Genetic functions are not necessarily continuous: for example the unit step function*

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

has the following[1 2] *genetic construction:*

$$f(x) := \left\{ \frac{x}{1+x^2} \right\} | \{\}.$$

In fact, genetic functions can be everywhere discontinuous:

Corollary 5. ω *is discontinuous everywhere with respect to the standard order topology.*

Proof. The underlying intuition here is that ω sends each x to its unique archimedean class. In particular, unless $x = y$, $|\omega^x - \omega^y| >> \mathbb{N}$. □

There is an immediate tension given the construction of genetic functions above with the previous construction of the genetic operations of addition, multiplication, and negation from

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[1] *This step function relies completely on simplicity to assign the values to 0 and 1 respectively.*

[2] *The uniformity of this definition is not proven in [3], but can be checked readily by the uniformity of the left option, e.g. that the rational function* $\frac{x}{1+x^2}$ *is an entire genetic function, and that the right options are vacuously uniform.*

the literature [1,5] et al, as these operations are used to define the Ring operations that characterize the surreal numbers, while also being claimed to be surreal functions proper in the sense of [3]. We resolve this tension as follows:

Definition 56. *Given genetic functions* f *and* g*,* f *is* $\leq_s$***-minimal*** *relative to* g *if for all* $\bar{x} \in \mathrm{No}$ *such that* $\mathsf{f}(\bar{x}) \neq \mathsf{g}(\bar{x})$*, we have* $\mathsf{f}(\bar{x}) <_s \mathsf{g}(\bar{x})$*.*

The definitions for addition and multiplication given in Conway, Gonshor, and elsewhere, are the $<_s$-minimal functions such that respectively addition is strictly increasing on both arguments, and multiplication satisfies the order property $xy + x'y' > x'y + xy'$ for all $x' < x$ and $y' < y$. Similarly, the ω function is the $\leq_s$ minimal functions such that $\omega^x > \omega^{x'}n$ for all $x' < x$ and $n \in \mathbb{N}$.

The tension noted above is obviated by first recognizing that every genetic function is an $\leq_s$-*minimal function* whose Left and Right options satisfy a theory for the corresponding function symbol. Then, the following proposition verifies that introducing new functions symbols to handle addition, negation, and multiplication will be equivalent to the Ring operations defined in [1,5].

Proposition 11. *The Ring operations* $+, \cdot,$ *and* $-$ *are genetic functions in the sense of [3]. Specifically, we identify addition with* $\sigma(x, y)$*, multiplication with* $\mu(x, y)$*, and negation with* $\nu(x)$*, and show that their construction in [1] agrees with their construction in [3]. That is, the Class of genetic functions generated from function symbols* $\{\mathrm{id}\}$ *is identical to the Class of genetic functions generated from* $\{\mathrm{id}, \nu, \sigma, \mu\}$*.*

Proof. For each of these functions, we run through the adjunction step starting with base ring of genetic functions $S = \{\mathsf{id}\}$ in the case of ν and σ. For the case of multiplication, we rely on the ring generated by $S = \{\mathsf{id}, \sigma, \nu\}$.

Negation

1. Let $S = \{\mathsf{id}\}$ and introduce $f \equiv \nu$. Then $K = S[v, w, \nu(v), \nu(w)] = \mathbb{Z}[v, w, \nu(v), \nu(w)]$.
2. In turn, $S(v, w) = \{c_1(c_2 x + c_3) + c_4 \colon c_1, c_2, c_3, c_4 \in K\}$ since $S = \{\mathsf{id}\}$. In particular, $\nu(v)$ and $\nu(w)$ are terms in K, so we may choose $c_4 = \nu(v)$ or $\nu(w)$ and set $c_1 = 0$ when forming subsets of $S(v, w)$. Straightforwardly of course, we note that we also have $-v, -w$ for free. The idea here is to show that we may introduce a function symbol for negation, and find that it agrees as we expect with the genetic definition of negation given in [1, 5].
3. Next, forming $R(v, w) = \mathrm{No}[S(v, w)]$, we note that $\nu(v), \nu(w)$ are still elements of $R(v, w)$.
4. We then choose $L_\nu = \{\nu(w)\}$ and $R_\nu = \{\nu(v)\}$.
5. Substituting x^L for v and x^R for w, and by induction supposing this is defined for all $y <_s x$, we obtain $L_\nu(x^L, x^R) = \{\nu(x^R)\}$ and $R_\nu(x^L, x^R) = \{\nu(x^L)\}$.
6. Finally, by induction have that $x^L < x^R$ implies $\nu(x^R) < \nu(x^L)$, so $\nu(x)$ will be defined.

We in turn find that $-x$ and $\nu(x)$ are identical by induction on the sets of options.

<u>Addition</u> We run through the steps as before, but the main thing to confirm is that we can construct the following two sets:

$$L_\sigma((x,y)^L,(x,y)^R) = \{\sigma(x^L,y),\sigma(x,y^L)\}$$

$$R_\sigma((x,y)^L,(x,y)^R) = \{\sigma(x^R,y),\sigma(x,y^R)\}$$

inside $S(v,w)$.

1. Let $S = \{id\}$ and introduce $f \equiv \sigma$. Then

$$K = S[v_1,v_2,w_1,w_2,\sigma(u_1,u_2)] = \mathbb{Z}[v_1,v_2,w_1,w_2,\sigma(u_1,u_2) : u_1 \in \{v_1,w_1\}, u_2 \in \{v_2,w_2\}]$$

2. In turn, $S(v,w) = \{c_1 + c_2(A\bar{x} + c_3) \colon c_1,c_2,c_3 \in K, A \in K^{1\times|x|}\}$ since $S = \{id\}$, and the identity function is taken to be a single variable function. In particular, $\sigma(v_1,v_2)$ and $\sigma(w_1,w_2)$ are terms in K, so we may set $c_2 = 0$ and $c_1 = \sigma(v_1,v_2)$ or $c_1 = \sigma(w_1,w_2)$.

3. Form $R(\bar{v},\bar{w}) = \mathrm{No}[S(\bar{v},\bar{w})]$.

4. We then choose $L_\sigma = \{\sigma(v_1,v_2)\}$ and $R_v = \{\sigma(w_1,w_2)\}$.

5. When substituting $\bar{x}^L$ for $\bar{v}$ and x^R for $\bar{w}$, we note that $\bar{x}^L \in L_{\bar{x}}$ is some tuple where at least one element y_i is a Left predecessor of the i^{th} (similarly for $R_{\bar{x}}$).

   Specifically, with $\bar{x} = \langle x,y\rangle$, we are ranging over generic $\langle x^L,y^L\rangle$, $\langle x^L,y\rangle$, and $\langle x,y^L\rangle$ (similarly for $R_{\bar{x}}$). In turn, this leads to $L_\sigma(\bar{x}^L,\bar{x}^R) = \{\sigma(x^L,y^L),\sigma(x^L,y),\sigma(x,y^L)\}$, and $R_\sigma(\bar{x}^L, \bar{x}^R) = \{\sigma(x^R, y^R), \sigma(x^R, y), \sigma(x, y^R)\}$. By an inductive cofinality and coinitiality arguments, we can further simplify this to

$$L_\sigma = \{\sigma(x^L, y), \sigma(x, y^L)\}$$

$$R_\sigma = \{\sigma(x^R, y), \sigma(x, y^R)\}$$

6. Finally, by induction have that $x^L + y < x + y < x^R + y$ (and similarly for other terms), which implies that $\sigma(x, y)$ will be defined on all of No.

We in turn identify that $x+y$ and $\sigma(x, y)$ are identical by induction on the sets of options.[1]

<u>Multiplication</u> It'll suffice to show that we can form

$$L_\mu(\bar{x}^L, \bar{x}^R) = \{\mu(x^L, y) + \mu(x, y^L) + (-\mu(x^L, y^L)), \mu(x^R, y) + \mu(x, y^R) + (-\mu(x^R, y^R))\}$$

$$R_\mu = \{\mu(x^L, y) + \mu(x, y^R) + (-\mu(x^L, y^R)), \mu(x^L, y) + \mu(x, y^R) + (-\mu(x^L, y^R))\}$$

with $S = \{id, +, -\}$, having identified $+, -$ with μ, ν. Running through the six steps as before

1. Set $S = \{id, \mu, \nu\}$ and substitute with $+, -$ as needed, and introduce $f \equiv \mu$. Then

$$K = S[v_1, v_2, w_1, w_2, \sigma(u_1, u_2)] = \mathbb{Z}[v_1, v_2, w_1, w_2, \sigma(u_1, u_2) : u_1 \in \{v_1, w_1\}, u_2 \in \{v_2, w_2\}]$$

---

[1] Of course, we also could have constructed this directly using the ring operations.

2. Let

$$S(v,w) = \{c_1 + c_2 h(A\bar{x} + c_3) \colon h \in \{id, \mu, \nu\}, c_1, c_2, c_3 \in K, A \in K^{n_h \times |x|}, \}$$

3. Form $R(\bar{v}, \bar{w}) = \text{No}[S(\bar{v}, \bar{w})]$.
4. Choose

$$L_\mu(\bar{v}, \bar{w}) = \{\mu(v_1, w_2) + \mu(w_1, v_2) - \mu(v_1, v_2), \mu(w_1, v_2) + \mu(v_1, w_2) - \mu(w_1, w_2)\}$$

and

$$R_\mu(\bar{v}, \bar{w}) = \{\mu(v_1, v_2) + \mu(w_1, w_2) - \mu(v_1, w_2), \mu(w_1, w_2) + \mu(w_1', w_2') - \mu(w_1, v_2)\}.$$

5. When substituting v_1, v_2, w_1, w_2, by cofinality and coinitiality form:

$$L_\mu = \{\mu(x^L, y) + \mu(x, y^L) - \mu(x^L, y^L), \mu(x^R, y) + \mu(x, y^R) - \mu(x^R, y^R)\}$$

and

$$R_\mu = \{\mu(x^L, y) + \mu(x, y^R) - \mu(x^L, y^R), \mu(x^R, y) + \mu(x, y^L) - \mu(x^R, y^L)\}.$$

6. The proofs of order preserving properties and by extension uniformity can be found in [1].

□

Remark 23. *We have shown that introducing new function symbols* σ, μ, ν *to handle the Ring structure will agree with the ordinary construction of the surreal numbers. We could also redefine genetic functions as being built first with respect to the Ring operations, and then for new function symbols. In any case, one major goal of this dissertation is to understand the substructures of the surreal numbers that are closed under some subring of genetic functions. This in turn requires that at a minimum we include addition and multiplication as generating functions.*

Corollary 6. *1. Any function symbol* f *of arity* $n \geq 2$ *and* $\bar{x} \in \mathrm{No}^n$ *such that the only terms in* L_f *are* $f(\bar{x})$ *where* $\bar{x}^L \in L_{\bar{x}}$ *(and similarly for* R_f*) is equivalent to addition of* n *summands.*

2. Any unary function symbol f *such that* $f(\nu) = \left\{f(x^R)\right\} | \left\{f(x^L)\right\}$ *will correspond to negation.*

Proof. 1. Let $\bar{x}_i^L$ denote the n-tuple of x_j such that for all $i \neq j$, $x_j' = x_j$ while $x_i^L <_s x_i$ and $x_i^L < x + i$. Similarly, let $\bar{x}_i^R$ denote the n-tuple such that $x_j^R = x_j$ for $j \neq i$ and $x_i^R <_s x_i$ and $x_i < x_i^R$. Suppose $L_f = \{f\}$ We can verify this via the adjunction argument for addition from the previous proposition,[1] although the main idea to note is that each $(\sum_{i \leq n} x_i)^L$ is cofinal with $f(x^L)$ (and similarly for $(\sum_{i \leq n} x_i)^R$).

[1]We could also show this by induction on n and the associativity of ordinary addition (see [1, 5] for further details of this proof).

To begin, for $\bar{x} = 0$, it's immediate that $f(0) = 0$ as $L_f(0) = R_f(0) = \emptyset$. By induction,

$$f(\bar{x}_i^L) = x_1 + \cdots + x_i^L + \cdots x_n$$

and similarly

$$f(\bar{x}_i^R) = x_1 + \cdots + x_i^R + \cdots + x_n$$

So for

$$f(\bar{x}) = \left\{f(\bar{x}_i^L)\right\} \mid \left\{f(\bar{x}_i^R)\right\} = \sum_{i \leq n} x_i.$$

2. Given the uniformity property we have $f^R(v) < f(v) < f^L(v)$, by $s-$minimality, $f(v)$ and $-v$ are both the simplest functions such that the Right options are less than the Left options. But then f is identical to negation.

□

3.2 Generations and ancestors

Before we cover several explicit examples of genetic functions, we will introduce the following concept to track the dependence of genetic functions on previously defined genetic functions in the adjunction and composition stages of forming $\mathcal{G}$.

Definition 57. *Let* $S_0 = \text{No}[\bar{x}]$*, and set* $S_0 = S^0$*. Then for every* $\alpha \in \text{On}$ *greater than 0, set*

$$S^\alpha = \bigcup_{\beta \in \alpha} S_0.$$

We then modify the adjunction stage for forming genetic functions as follows:

First, form

$$K^{\alpha} := \mathrm{No}[\{g(\bar{u}), g(\bar{v}) \colon g \in (S^{\alpha} \cup \{f_n \colon n \in \mathbb{N} \text{ where } f_n \text{ is a function symbol of } n\text{-arity})\}].$$

Then form

$$S^{\alpha}(\bar{u}, \bar{v}) := \{c_1 + c_2 h(\bar{c_3} \cdot \bar{x} + \bar{c_4}) \colon c_1, c_2, \bar{c}_3, \bar{c}_4 \in K^{\alpha}, h \in S^{\alpha}\}$$

where $\bar{x}, \bar{c_3}$, *and* $\bar{c_4}$ *respects the arity of* h, *and further, each* c_i *term respects the arity of the terms formed with respect to the adjunction of* $g(\bar{u}), g(\bar{v})$ *terms.*

Afterwards, we form

$$R^{\alpha}(u, v) := \mathrm{No}[S^{\alpha}(\bar{u}, \bar{v})]_{P_{\alpha}},$$

where P_{α} *are the strictly positive terms in* $\mathrm{No}[S^{\alpha}(\bar{u}, \bar{v})]$ *consisting solely of function symbols in* S^{α}, *i.e. we introduce entire rational genetic functions defined from earlier* generations *of genetic functions.*

We now introduce three new steps: Denote by

$$S^{\alpha}_{\mathcal{F}} = \{f \in \mathcal{G} \colon L_f, R_f \subset R^{\alpha}(u, v)\},$$

and letting $\langle S \rangle$ *denote the closure of a set of functions* S *under composition, we finally set* $S_{\alpha} = \mathrm{No}[\langle S^{\alpha} \cup \langle S^{\alpha}_{\mathcal{F}} \rangle]$. *Then, we properly define* ***generation*** *as the function* $\daleth : \mathcal{G} \to \mathrm{On}$ *such that* $\daleth(f)$ *sends* f *to the least ordinal* α *such that* $g \in S_{\alpha}$.

For each $\mathsf{f} \in \mathcal{G}$ *such that* $\daleth(\mathsf{f}) = \alpha$*, let* $\mathcal{A}_\mathsf{f} \subset S_\alpha$ *denote the set of* ***ancestors*** *of* f*, defined to be the smallest set of all function symbols closed under composition, such that for*

$$K_\mathsf{f} := \mathrm{No}[\{g(\bar{u}), g(\bar{v})\colon g \in \mathcal{A}_\mathsf{f} \cup \{\mathsf{f}\}\}]$$

and

$$S_\mathsf{f}(\bar{u}, \bar{v}) := \{c_1 + c_2 h(c_3 \cdot x + c_4)\colon c_1, c_2, c_3, c_4 \in K_\mathsf{f}, h \in \mathcal{A}_\mathsf{f}\},$$

we have $L_\mathsf{f}, R_\mathsf{f} \subset \mathrm{No}[S_\mathsf{f}(\bar{u}, \bar{v})]$*.*

Further, we can extend $\daleth$ *to range over sets of genetic functions* $\mathcal{G}$*, with* $\daleth(\mathcal{G}) = \max\{\daleth(g)\colon g \in \mathcal{G}\}$*. Finally, given any set* $\mathcal{G}$*, we set*

$$\mathcal{G}^* = \mathbb{Z}\left[\left\langle \bigcup_{g \in \mathcal{G}} \mathcal{A}_g \cup \{g\} \right\rangle\right],$$

so that $\mathcal{G}^*$ *denotes the polynomial ring of terms formed by adjoining the closure under composition of the set-union of* $\mathcal{G}$ *and each ancestor of* $g \in \mathcal{G}$*.*

There are several properties of $\daleth$ we need to verify:

Proposition 12. *For every* $\mathsf{f}, g \in \mathcal{G}$ *and* $\mathcal{G} \subseteq \mathcal{G}$*,*

1. $\daleth(\mathsf{f} + g) \le \max\{\daleth(\mathsf{f}), \daleth(g)\}$*;*
2. $\daleth(\mathsf{f}g) \le \max\{\daleth(\mathsf{f}), \daleth(g)\}$*;*
3. $\daleth(\mathsf{f} \circ g) \le \max\{\daleth(\mathsf{f}), \daleth(g)\}$*;*

4. $\daleth(\mathcal{G}^*) = \daleth(\mathcal{G})$.

Proof. 1. Since $S_\beta \subseteq S_\alpha$ for all $\beta \in \alpha$, we may without loss of generality suppose that $\daleth(f) = \daleth(g) = \alpha$. It is immediate that $f + g \in S_\alpha$, since $S_\alpha := \mathrm{No}[\langle S^\alpha \cup S^\alpha_{\mathcal{F}} \rangle]$.

2. We similarly find that $\daleth fg \leq \max\{\daleth f, \daleth g\}$, whenever $f, g \in S_\alpha$, where $\alpha = \max\{\daleth f, \daleth g\}$.

3. We similarly find that $\daleth(f \circ g) \leq \max\{\daleth f, \daleth g\}$, whenever $f, g \in S_\alpha$, where $\alpha = \max\{\daleth f, \daleth g\}$.

4. This follows by application of 1.-3, since

$$\mathcal{G}^* = \mathbb{Z}\left[\left\langle \bigcup_{g \in \mathcal{G}} \mathcal{A}_g \cup \{g\} \right\rangle\right] \subset \mathrm{No}[\langle S^\alpha \cup S^\alpha_{\mathcal{F}} \rangle] = S_\alpha$$

□

CHAPTER 4

PSEUDO-ABSOLUTE VALUES AND DESCRIPTIVE FUNCTIONS

4.1 Some results characterizing α_0

We first introduce the following definition:

Definition 58. *For each surreal number* $a = \sum_{\nu a} \omega^{a_i} r_i$*, we let* $\sum_{i<n_a} \omega^{a_i} r_i$ *denote the* ***ordinal head*** *of* a*. That is, we set* n_a *to denote the number of initial summands of the Conway normal form of a surreal number* a *that contribute to* $\alpha_0(a)$*, so that the ordinal head is the maximal truncation of the surreal number* a *that is an ordinal.*

Remark 24. *Since each* $\alpha_0(a)$ *is an ordinal with a finite length Cantor normal form, it follows that* $n_a \in \omega$ *for all* $a \in \text{No}$*.*

Theorem 43. *For all* $a \in \text{No}$*, with Conway normal form* $\sum_{\nu a} \omega^{a_i} r_i$*, let* $(\bar{a}, \bar{r})$ *be the finite sequence of pairs of length* $n_a + 1$ *such that* (a_i) *are the descending exponents and* (r_i) *are the coefficients from the initial terms of the Conway normal form of a surreal number* a *such that for all* $i < n_a$*,* $(a_i, r_i) \in \text{On} \times \mathbb{Z}^+$*, and for* $i = n_a$*,* $(a_i, r_i) \in \text{No} \times \mathbb{R}^\times \setminus (\text{On} \times \mathbb{Z}^+)$*. Then*

$$\alpha_0(a) = \begin{cases} \sum_{i<n_a} \omega^{a_i} r_i + \omega^{\alpha_0(a_{n_a})} & a_{n_a} \in \text{No} \setminus \text{On} \wedge r_{n_a} > 0 \\ \sum_{i<n_a} \omega^{a_i} r_i + \omega^{a_{n_a}} \alpha_0(r_{n_a}) & a_{n_a} \in \text{On} \wedge r_{n_a} \in \mathbb{R}_{>0} \setminus \mathbb{Z}^+ \\ \sum_{i<n_a} \omega^{a_i} r_i & r_{n_a} < 0 \end{cases}$$

Proof. This immediately follows by the definition of α_0 as the ordinal number of $\oplus$ symbols in the first pair of sign symbols in the sign sequence of a. □

Corollary 7. *For all* $a, b \in \mathrm{No}$*, we have*

$$\sum_{i<n_a} \omega^{a_i} r_i + \sum_{j<n_b} \omega^{b_j} s_j \leq \alpha_0(a+b)$$

We will need some of the results to prove facts about intervals of reduction:

Lemma 8. *For all* $r, s \in \mathbb{R}_{>0}$*,* $\alpha_0(r+s) \leq \alpha_0(r) + \alpha_0(s)$*.*

Proof. This follows immediately after observing that for all positive non-integral reals, $\alpha_0(r) = \lfloor r \rfloor + 1 = \lceil r \rceil$, and otherwise, $\alpha_0(r) = \lfloor r \rfloor = \lceil r \rceil$. From here, we note that

$$\lceil r+s \rceil \leq \lceil r \rceil + \lceil s \rceil$$

□

Theorem 44 (Properties of α_0)**.** *Below are a few properties of* α_0*:*

1. *if* $\phi(a) = 1$ *and* $\beta_0(a) = 0$*, then* $\alpha_0(a) = a$ *is an ordinal;*
2. α_0 *is idempotent;*
3. *For all* $a \in \mathrm{No}$*,* $\alpha_0(a+1) \leq \alpha_0(a) + 1$*;*
4. $\alpha_0(a+b) \leq \alpha_0(a) + \alpha_0(b)$*;*

Proof. 1. This is immediate, as $\alpha_0 : \mathrm{No} \to \mathrm{On}$;

2. Following (1), this is immediate, as $\alpha_0 \restriction_{\mathrm{On}} = \mathrm{id}_{\mathrm{On}}$;

3. If $a \leq 0$, then $\alpha_0(a+1) \leq 1 = 0 + 1 = \alpha_0(a) + 1$. If $a > 0$, then either

$$(a+1) = \langle \alpha_0(a), \beta_0(a) \rangle \frown \text{o.p}$$

   if for all $i < n_a$, $a_i > 0$, and otherwise

$$(a+1) = \langle \alpha_0(a) + 1, \beta_0(a) \rangle \frown \text{o.p.}$$

   so $\alpha_0(a+1) \leq \alpha_0(a) + 1$;

4. If $a \geq 0$ and $b \leq 0$, then this is immediate by Theorem 43, so without loss of generality, suppose that $a, b > 0$ and furthermore that $a_{n_a} \geq b_{n_b}$. Then, after expanding the Conway normal form, we have

$$\begin{aligned} a + b &= \sum_{\nu a} \omega^{a_i} r_i + \sum_{\nu b} \omega^{b_j} s_j \\ &= \left(\sum_{i < n_a} \omega^{a_i} r_i + \sum_{j < n_b} \omega^{b_j} s_j \right) + \omega^{a_{n_a}} r_{n_a} + \omega^{b_{n_b}} s_{n_b} + \text{l.t.} \end{aligned}$$

   We now proceed by cases. First, suppose that $r_{n_a}, s_{n_b} > 0$. Then,

   - If $a_{n_a} = b_{n_b}$, and $a_{n_a} \in \mathrm{No} \backslash \mathrm{On}$, this result is immediate.
   - If $a_{n_a} = b_{n_b} \in \mathrm{On}$, then by Theorem 43, $r_{n_a}, s_{n_b} \in \mathbb{R}_{>0} \backslash \mathbb{Z}^+$. We now use Lemma 8, to reach our desired conclusion.

- If $a_{n_a} > b_{n_b}$, then we have

$$\alpha_0(a+b) = \left(\sum_{i<n_a} \omega^{a_i} r_i + \sum_{j<n_b} \omega^{b_j} s_j\right) + \omega^{\alpha_0(a_{n_a})} \leq \alpha_0(a) + \alpha_0(b)$$

if $a_{n_a} \in \text{No}\backslash\text{On}$, and otherwise

$$\alpha_0(a+b) = \left(\sum_{i<n_a} \omega^{a_i} r_i + \sum_{j<n_b} \omega^{b_j} s_j\right) + \omega^{a_{n_a}} \alpha_0(r_{n_a}) \leq \alpha_0(a) + \alpha_0(b)$$

Next, if both r_{n_a} and s_{n_b} are negative real numbers, then we may stop. We conclude by checking the following cases:

- $r_{n_a} > 0 > s_{n_b}$ and $r_{n_a} + s_{n_b} > 0$:
  - If $a_{n_b} = b_{n_b} \in \text{No}\backslash\text{On}$, then the result is immediate with equality holding.
  - If $a_{n_b} = b_{n_b} \in \text{On}$, then we have $\alpha_0(r_{n_a} + s_{n_b}) \leq \alpha_0(r_{n_a})$.
    * If $r_{n_a} + s_{n_b} \in \mathbb{Z}^+$, since $r_{n_a} + s_{n_b} < r_{n_a}$ and $\alpha_0(r_{n_a} + s_{n_b}) = r_{n_a} + s_{n_b}$, we have $\alpha_0(r_{n_a}) = \alpha_0(r_{n_a} + s_{n_b}) + 1$. In this specific subcase, we have

$$\alpha_0(a+b) = \sum_{i<n_a} \omega^{a_i} r_i + \sum_{j<n_b} (\omega^{b_j} s_j + \omega^{r_{n_a}})(r_{n_a} + s_{n_b}) + \omega^{c_{k_0}} t_{k_0} + \cdots + \omega^{c_{k_n}} t_{k_n} + \text{l.t.}$$

for a finite sequence of descending $c_{k_m} \in \{a_i \mid n_a < i < \nu a\} \cup \{b_j \mid n_b < j < \nu b\}$, and similarly $r_{k_m} \in \{r_i \mid n_a < i < \nu a\} \cup \{b_j \mid n_b < j < \nu b\}$ satisfying

Theorem 43. No matter what this subsequent sequence consists of, because $\alpha_0(r_{n_a}) > \alpha_0(r_{n_a} + s_{n_b})$, we find

$$\alpha_0(a+b) \leq \alpha_0(a) + \alpha_0(b),$$

because

$$\omega^{a_{n_a}}\alpha_0(r_{n_a}) = \omega^{a_{n_a}}(\alpha_0(r_{n_a}+s_{n_b})+1) > \omega^{a_{n_a}}\alpha_0(r_{n_a}+s_{n_b})+\sum_{m<n}\omega^{c_k}t_k+\text{f.t.}$$

as each c_k is some ordinal below a_{n_b}.

* Otherwise, for $r_{n_b} + s_{n_a} \in \mathbb{R}_{>0}\backslash\mathbb{Z}^+$, we find that $\alpha_0(r_{n_a} + s_{n_b}) \leq \alpha_0(r_{n_a})$, and the inequality follows as desired.

- $r_{n_a} > 0 > s_{n_b}$ and $r_{n_a} + s_{n_b} \leq 0$. If this is the case, then our inequality follows as desired by the same reasoning as in the previous bullet points.

Combining Corollary 7 and Theorem 44, we now have a tight characterization of the ordinal interval the sum of any two surreal numbers will lie in, namely:

Corollary 8. *For all* $a, b \in \text{No}$*, we find that*

$$\alpha_0(a+b) \in [\sum_{i<n_a}\omega^{a_i}r_i + \sum_{j<n_b}\omega^{b_j}s_j, \alpha_0(a) + \alpha_0(b)]$$

□

Our focus on α_0 is essential for understanding the conditions under which *reduction*, to be introduced in the following section, succeeeds and fails under translation. Towards that end, the following theorem fully Classifies all conditions under which $\alpha_0(a+b) = \alpha_0(a) + \alpha_0(b)$.

Theorem 45. *1. If* $a, b < 0$*, then* $\alpha_0(a+b) = \alpha_0(a) + \alpha_0(b) = \alpha_0(a) = 0$*;*

2. If $a \geq b \geq 0$*, then* $\alpha_0(a+b) = \alpha_0(a) + \alpha_0(b)$ *if and only if*

(a)

$$((r_{n_a} > 0 \vee s_{n_b} > 0) \rightarrow$$

$$(a_{n_a} = b_{n_b} \in \mathrm{On} \wedge ((r_{n_a} > 0 \wedge s_{n_b} > 0 \rightarrow (\lfloor r_{n_a} + s_{n_b} \rfloor > \lfloor r_{n_a} \rfloor + \lfloor s_{n_b} \rfloor)$$

$$\wedge (s_{n_b} < 0 \rightarrow r_{n_b} + s_{n_b} > \lfloor r_{n_a} \rfloor) \wedge (r_{n_a} < 0 \rightarrow r_{n_a} + s_{n_b} > \lfloor s_{n_b} \rfloor))))$$

AND

(b)

$$(r_{n_a} \leq 0 \wedge s_{n_b} \leq 0) \rightarrow$$

$$((r_{n_a} = s_{n_a} = 0) \vee (r_{n_a} < 0 \rightarrow a_{n_a} < b_{n_b - 1}) \vee (s_{n_b} < 0 \rightarrow b_{n_b} < a_{n_a - 1})).$$

3. If $a > 0 \geq b$*, then* $\alpha_0(a+b) = \alpha_0(a)$ *if and only if*

$$a_{n_a} \geq b_0 \wedge (b_0 = a_{n_a} \rightarrow (r_{n_a} > 0 \rightarrow$$

$$(a_{n_a} \in \mathrm{On} \rightarrow s_0 + r_{n_a} > \lfloor r_{n_a} \rfloor) \wedge (a_{n_a} \in \mathrm{No} \backslash \mathrm{On} \rightarrow r_{n_a} + s_0 > 0)))$$

4. *Suppose that* $a > b > 0$. *Then* $\alpha_0(a + b) = \alpha_0(a)$ *if and only if* $a_{n_a} \geq b_0$ *and*

$$a_{n_a} = b_0 \to (a_{n_a} \in \text{No}\backslash\text{On} \vee r_{n_a} < 0 \wedge r_{n_a} + s_0 \leq 0) \vee (r_{n_a} > 0 \wedge \alpha_0(r_{n_a} + s_0) = \alpha_0(r_{n_a}))$$

Proof. 1. This follows immediately, as $\alpha_0(a) = 0 \iff a \leq 0$.

2. In the converse direction, first if $r_{n_a}, s_{n_b} < 0$, then this follows directly by Theorem 43. Additionally, if $r_{n_a} = 0$ or $s_{n_b} = 0$, then we have an ordinal, and so the rest of (b) follow directly from Theorem 43.

   Now supposing that the condition for (a) holds and we have $r_{n_a} > 0$ or $s_{n_b} > 0$, then in a straightforward manner we apply Theorem 43 (in particular the results regarding $\alpha_0(r_{n_a}) + \alpha_0(s_{n_b}) = \alpha_0(r_{n_a} + s_{n_b})$.) In each case, we have addition on the ordinal heads of a and b so that $\alpha_0(a + b) = \alpha_0(a) + \alpha_0(b)$.

   To prove the forward direction, we will prove the contrapositive statement. Without loss of generality, we may suppose that $r_{n_a} > 0$ or $s_{n_b} > 0$, and that $a_{n_a} = b_{n_b} \in \text{On}$. From here, it will suffice to show for $a_{n_a} = b_{n_b}$ and $r_{n_a}, s_{n_b} > 0$ such that $\lfloor r_{n_a} + s_{n_b} \rfloor \leq \lfloor r_{n_a} \rfloor + \lfloor s_{n_b} \rfloor$ that $\alpha_0(r_{n_a} + s_{n_b}) < \alpha_0(r_{n_a}) + \alpha_0(s_{n_b})$. But this is precisely what happens when $\lfloor r_{n_a} + s_{n_b} \rfloor \leq \lfloor r_{n_a} \rfloor + \lfloor s_{n_b} \rfloor$, as for $r \in \mathbb{R}_{>0}\backslash\mathbb{Z}$, $\alpha_0(r) = \lfloor r \rfloor + 1 = \lceil r \rceil$.

3. We prove this in a manner similar to part 2. In the converse direction, we apply Theorem 43. In the forward direction, we prove the contrapositive by applying Theorem 43 to the two cases where $a_{n_a} = b_0$, $r_{n_a} > 0$, and either $a_{n_a} \in \text{On}$ such that $s_0 + r_{n_a} \leq \lfloor r_{n_a} \rfloor$, or $a_{n_a} \in \text{No}\backslash\text{On}$ and $r_{n_a} + s_0 \leq 0$.

4. As before, we prove the converse and the contrapositive, and in both cases we apply Theorem 43 to find the desired result.

□

Next, we observe that just as the initial ordinal heads of surreal numbers obey an additive inequality, so do the cumulative sums of α_i:

Theorem 46. *Recall that* $\gamma_\mu(a) = \bigoplus_{i \le \mu} \alpha_\mu(a)$. *For all ordinals* $\mu \in \text{On}$ *and all* $a, b \in \text{No}$*,*

$$\gamma_\mu(a + b) \le \gamma_\mu(a) + \gamma_\mu(b).$$

Proof. First, we note that for all $a, b \in \text{No}$, $(a+b)^+ \le a^+ + b^+$, so for all $\mu \ge \max\{\phi(a), \phi(b), \phi(a+b)\}$, we have the desired inequality. The proof of this fact follows by induction, as described in Lemma 3.4 in [2], and given that $\alpha : \text{No} \to (\text{On} \to \text{On})$ is defined on all μ and for all $a, b \in \text{No}$ as an eventually zero function.

We proceed by induction on $\iota(a) + \iota(b)$.

When $\iota(a) + \iota(b) = 0$, the result is immediate for all $\mu \in \text{On}$. Now supposing that the inequality holds for all $\mu \in \text{On}$ for all $(a, b) \in \text{No} \times \text{No}$ such that $\iota(a) + \iota(b) < \lambda$. Choose $(a, b) \in \text{No} \times \text{No}$ such that $\iota(a) + \iota(b) = \lambda$. By Theorem 44.4, we have that

$$\gamma_0(a + b) = \alpha_0(a + b) \le \alpha_0(a) + \alpha_0(b) = \gamma_0(a) + \gamma_0(b).$$

Now suppose that the result holds for all $\mu < \xi$. Note that if $\xi \geq \lambda \geq \phi(a) + \phi(b)$, by Lemma 3.4.1 in [2], the result holds, so without loss of further generality, we may suppose that $\xi < \lambda$. In any case, because

$$a + b = \left\{a^L + b, a + b^L\right\} \mid \left\{a^R + b, a + b^R\right\}$$

we can apply our induction hypothesis to the predecessors of a^L (resp. a^R, b^L, and b^R) corresponding to proper restrictions of a to the first ξ many pairs, from which we derive in the case of a^L (we may apply the induction hypothesis because $\iota(a^L) + \iota(b) < \lambda$),

$$\gamma_\xi(a + b) = \gamma_\xi(a^L + b) \leq \gamma_\xi(a^L) + \gamma_\xi(b) = \gamma_\xi(a) + \gamma_\xi(b).$$

□

4.2 The $\surd$ pseudo-absolute value

In this section we introduce the *surd* pseudo-absolute value for measuring the growth of complexity of surreal numbers. While other authors have called $\surd$ the degree or logarithm of an ordinal, we prefer calling this the "surd", if only because that is the name of the LaTeXcommand, and also because it can be considered a portmanteau of surreal degree.

Before defining $\surd$ and verifying that it satisfies the requirements for a pseudo-absolute values, we remind the reader of three subclasses of limit ordinals.

Definition 59. *Following Hessenburg,*

- γ *ordinals* ***additively indecomposable****, i.e. for all* $x, y \in \gamma$*,* $x \oplus y \in \gamma$*.*

- δ *ordinals are additionally* ***multiplicatively indecomposable****, i.e. for all* $x, y \in \delta$, $x \otimes y \in \delta$.
- ***Epsilon*** *ordinals* ϵ *are ordinals ordinals* $\varepsilon \ni 1$ *such that for all* $x \in \epsilon$, $x^{\epsilon} = \epsilon$.

Further, recalling that every ordinal α has a Cantor normal form $\sum\limits_{N\alpha} \omega^{\alpha_i}$ where $\alpha_i \geq \alpha_j$ for all $i < j \in N\alpha$, and $N\alpha$ is a natural number, we define the following equivalence relation:

Definition 60. *Let* $\alpha_1, \alpha_2 \in \text{On}$ *have Cantor normal form* $\sum\limits_{j \in n_i} \omega^{\alpha_{i,j}}$ *for* $i = 1, 2$. *Then say* $\alpha_1 \sim_\Gamma \alpha_2$ *if and only if* $\alpha_{1,0} = \alpha_{2,0}$.

It is immediate that this is an equivalence relation, and that it identifies numbers by the leading exponent of the Cantor normal form a surreal number's length. We note that in turn each equivalence Class has a unique minimal element, namely the gamma number in the infinite case, 1 in the finite case, and 0 otherwise.

Remark 25. *While* $\sim_\Gamma$ *is an equivalence relation put on ordinals, each surreal number* a *is associated with some equivalence Class* $[\iota a]_{\sim_\Gamma}$. *Moreover, each of these equivalence Classes has a* simplest *element, in the sense of [17], [9]. Namely, the simplest element is the* Γ*-ordinal* ω^α *such that* $\omega^\alpha \sim_\Gamma \iota a$. *We will denote this by* $\sqrt{\iota a}$, *and for general* $\alpha \in \text{On}$, *by* $\sqrt{\alpha}$. *In turn, the simplest element of* $[a]_{\sim_\Gamma}$ *is also the minimal element of* $[a]_{\sim_\Gamma}$. *Finally, we may regard* $\sqrt{}$ *as a Class function from the Class of ordinals to the Class of* $\Gamma-$*ordinals.*

The following are some immediate lemma concerning $\sqrt{}$:

Lemma 9. *For all* $a, b \in \mathrm{No}$,

$$\sqrt{\iota}(a+b) \leq \max\{\sqrt{\iota}a, \sqrt{\iota}b\}.$$

Proof. First, $\iota(a+b) \leq \iota(a) + \iota(b)$, from which we derive

$$\sqrt{(}\iota(a+b)) \leq \sqrt{(}\iota a + \iota b).$$

We see that $\sqrt{(}\iota a + \iota(b)) = \max\{\sqrt{(}\iota a), \sqrt{(}\iota b)\}$ follows because $\sqrt{(}\iota a + \iota b)$ is the leading term of the Cantor normal form of $\iota a + \iota b$, which by the descending order of the Cantor normal form must belong to either ιa or ιb, which in turn is $\sqrt{\iota}a$ or $\sqrt{\iota}b$. □

Remark 26. *The argument above, and throughout this dissertation, uses the convention that the Cantor normal form of an ordinal* α *be written in terms* ω^{α_i} *such that* $\alpha_i \geq \alpha_{i+1}$ *for* $i \in \mathsf{N}\alpha$, *where* $\mathsf{N}\alpha \in \omega$ *denotes the number of summands in the normal form. These arguments can be modified to handle the alternate conventions where the Cantor normal form consists of terms* $\omega^{\alpha_i} n_i$ *such that* $n_i \in \omega$ *and* $\alpha_i > \alpha_{i+1}$. *However, it is more convenient to use the former convention, especially when we partition the sign sequence into the constituent parts contributing to each summand. For example,* $\omega\frac{1}{2}$ *has the sign sequence* $\langle \omega, \omega \rangle$. *It has length* $\omega 2$, *but when analysing the sequence, we may find it useful to first note that* ω *contributes to the first summand of the Cantor normal form, and that the coefficient* $\frac{1}{2}$ *contributes to the second term. In contrast,* $\omega 2 + \pi$ *has length* $\omega 3$, *and we find that the final term in the first* *convention is contributed entirely by* π*. In the latter convention, we would need to decompose the* n_i *term, as opposed to examining the* i^{th} *term of the sum.*

We further make the following observation regarding $\surd\iota$:

Theorem 47. $\surd : \mathrm{On} \to \Gamma''\mathrm{On} \cup \{0, 1\}$ *is an extended absolute value, namely:*

1. $\surd x = 0 \iff x = 0$*;*
2. $\surd xy = \surd x \surd y$*;*
3. *the triangle inequality holds.*

Proof. 1. In the reverse direction, if $x = 0$, then $\surd 0 = 0$. In the forward direction, if $\surd x = 0$, then the leading term of the Cantor normal form is 0. But then $x = 0$ since the exponent of the terms of the Cantor normal form are ω^{α_i} such that $\alpha_i \geq \alpha_{i+1}$ or 0.

2. Let $x = \sum \omega^{\alpha_i}$ and $y = \sum \omega^{\beta_j}$. Then $\surd xy = \omega^{\alpha_0 + \beta_0} = \omega^{\alpha_0}\omega^{\beta_0} = \surd x \surd y$.

3. This follows by weakening Lemma 9.

□

It remains to be shown that we can extend $\surd$ to an extended absolute value on the surreals by way of $\surd\iota$. Conditions (1) and (3) are immediate, and so it can be extended to a quasi-metric. What would need to be shown is that $\surd\iota(xy) = \surd\iota x \surd\iota y$ for all $x, y \in \mathrm{No}$. By Theorem 47, this follows immediately whenever $\iota(xy) = \iota x \iota(y)$. However, this will not hold in general. Consider reciprocals, $x = \omega$ and $y = \omega^{-1}$. It is known that $\iota x = \omega$ and $\iota y = \omega$, while $\iota xy = 1$. So we have $\surd\iota xy = 1 < \omega^2 = \surd\iota x \surd\iota y$. So it is clear that $\surd$ can't be extended to an absolute value on No by pre-composing with ι. However, we may be able to prove that it can be extended to a pseudo-absolute value, i.e. that $\sqrt{\iota(xy)} \leq \sqrt{\iota x}\sqrt{\iota y}$, and from there, we can use that to study the complexity of surreal-valued functions. Towards this end, we will need several results:

Theorem 48. *1. For all* $a \in \mathrm{No}$*, there is no pair* $b, c \in \mathrm{No}$ *such that* $\sqrt{\iota b}, \sqrt{\iota c} < \sqrt{\iota a}$*, and* $b + c = a$.

2. For all $a \in \mathrm{No}$*, if* $\sqrt{\iota(a)} \in \Delta''\mathrm{On}$*, then there is no pair* $b, c \in \mathrm{No}$ *such that* $\sqrt{\iota b}, \sqrt{\iota c} < \sqrt{\iota a}$*, and* $bc = a$.

Proof. 1. This follows immediately from Lemma 9, since $\sqrt{\iota a} < \sqrt{\iota a}$ otherwise.

2. Suppose $\sqrt{\iota a}$ that is a delta number and $b, c \in \mathrm{No}$ are such that $\sqrt{\iota b}, \sqrt{\iota c} < \sqrt{\iota a}$. Recall the *weak product lemma*, Corollary 4.3 of [2], for any two surreal numbers b, c,

$$\iota(bc) \leq \omega\iota(b)^2\iota(c)^2.$$

Hence,

$$\iota bc \leq \omega(\iota b)^2(\iota c)^2 < \sqrt{\iota a} \leq \iota a,$$

as $\sqrt{\iota a}$ is multiplicatively indecomposable. But since since $\iota bc < \iota a$, no such b, c will satisfy $bc = a$.

□

We recall Lemma 4.1 from [2]. Specifically, for all $a \in \mathrm{No}$, $\iota(a) \leq \iota(\omega^a) \leq \omega^{\iota a}$. In conjunction with Theorem 48, we have the following corollary:

Corollary 9. *For all* $a, b, c \in \text{No}$*, if* $\sqrt{\iota} b < \sqrt{\iota} a$ *and* $\sqrt{\iota} c < \sqrt{\iota} a$*, then* $\sqrt{\iota}\omega(b + c) < \omega(\iota a)$*.*

Proof. First, note that $\sqrt{\omega \iota a} = \omega \sqrt{\iota a}$ for all a. The rest is a straightforward result that follows by applying Theorem 48.1 to Lemma 4.1 from [2]. Specifically,

$$\sqrt{\iota}\omega^{b+c} \overset{\text{Lem4.1}}{\leq} \sqrt{\omega^{\iota(b+c)}} = \omega^{\sqrt{\iota} b+c} \overset{\text{Thm48.1}}{<} \omega^{\sqrt{\iota} a}.$$

□

Theorem 49. $\sqrt{\iota}$ *is a pseudo-absolute value on* No*. Specifically,* $\sqrt{\iota}(ab) \leq \sqrt{\iota}(a)\sqrt{\iota}(b)$ *for all* $a, b \in \text{No}$*.*

Proof. First,

$$\begin{aligned}
\sqrt{\iota}(ab) &= \sqrt{\iota}\left((\sum_{\nu a} \omega^{a_i} r_i)(\sum_{\nu b} \omega^{b_j} s_j) \right) \\
&= \sqrt{\iota}\left(\sum_{\nu(ab)} \omega^{(ab)_k} t_k \right) \\
&= \sqrt{\left(\bigoplus_{\nu(ab)} \iota(\omega^{(ab)_k^{o}} \oplus \omega^{(ab)_k^{+}})\iota(t_k^{\flat}) \right)} \\
&= \omega^{\zeta}
\end{aligned}$$

where $\zeta \in \text{On}$. Then, since we only ever have finitely many pairs (i, j) such that $a_i + b_j = c$ and $\nu ab \leq n u a \nu b$, without loss of generality we may suppose that $\nu ab = \nu a \times \nu b$ so that there is a one-to-one correspondence between k and (i, j), and thus we can assume there is an order mapping between k and (i,j) pairs so that

$$\omega^\zeta = \sqrt{\left(\bigoplus_{\nu(ab)} \iota(\omega^{(ab)_k^o}) \oplus \omega^{(ab)_k^o}\iota(t_k^\flat))\right)} = \sqrt{\left(\bigoplus_{\nu a \times \nu b} \iota(\omega^{(a_i+b_j)^o}) \oplus \omega^{(a_i+b_j)^+}\iota((r_i s_j)^\flat)\right)}.$$

In cases where we don't necessarily have the order mapping, we note that there is always going to be a finite-to-one map of pairs (i, j) to a corresponding k, and the finite product of reals $\prod_{a_i+b_j=c_k} r_i s_j = t_k$.

From here we consider the following four cases:

1. we have a head composed of $\omega^\zeta \leq \nu ab$ terms, i.e. a head composed of ω^ζ many terms each of length less than ω^ζ contributes the $\sqrt{\iota(ab)}$ value;

2. we have a head composed of some limit ordinal many terms $< \omega^\zeta$ where each term is of length less than ω^ζ;[1]

3. If not (1) nor (2), then ω^ζ is contributed to by some β value appearing in the sign sequence of $(\omega^{(ab)_k^o})$;

4. If not (1) nor (2) nor (3), then ω^ζ is contributed by some $\omega^{(ab)_k^+}\iota(t_k)$ value;

We make the following claim:

Claim 1. *These cases are exhaustive.*

---

[1]For example, say $a = \sum_{n\in\omega} \omega^{\frac{3}{2^{n+1}}}$. Then $(a) = \langle\omega^2, \omega^3\rangle \frown \langle\omega, \omega^2\rangle \frown\frown_\omega \langle\omega^2 + \omega, \omega^2\rangle$, whence $\iota(a) = \sqrt{\iota(a)} = \omega^3$.

Proof. (of claim) Supposing that $\sqrt{\iota(ab)} = \omega^\zeta$ but ω^ζ does not arise from any of the four conditions listed above, i.e., $\omega^\zeta > \nu ab$, but no term is greater than ω^ζ. But this is impossible since ω^ζ is a gamma number. On the other hand, if $\omega^\zeta \leq \nu ab$ and one of the terms is at least length $\geq \omega^\zeta$, then one of conditions 3 or 4 must apply.

Specifically, if there were some reduced summand of length $\geq \omega^\zeta$, we may take the least-indexed summand due to the well-ordering of our index. But then since the length of this summand is $\geq \omega^\zeta$ either there is no β term such that $\sqrt{}(\omega^{\gamma_l(ab)^o_k+1}\beta_l((ab)^o_k)) = \omega^\zeta$, in which case $(ab)^+_k = \zeta$, i.e (4) holds. If (3) and (4) both fail, then the length is less than ω^ζ. □

Now we verify the theorem in order of the cases:

1. Supposing that $\sqrt{\iota(ab)} = \omega^\zeta \leq \nu ab$, then because $\nu ab \leq \nu a \times \nu b \leq \iota a \iota b$, whence $\omega^\zeta \leq \iota a \iota b$. Then because $\sqrt{}$ is a monotonic mapping[1] such that $\sqrt{}\sqrt{} = 1_{\{0,1\}\cup\Gamma''\mathrm{On}}$, we find that $\omega^\zeta \leq \sqrt{\iota a}\sqrt{\iota b}$ by Theorem 47. Thus $\sqrt{\iota(ab)} \leq \sqrt{\iota a}\sqrt{\iota b}$.

2. Supposing that we have a limit ordinal number of cases less than $\omega^\zeta \leq \nu ab$ such that the sum of reduced lengths of each term is ω^ζ, the same argument as in case (1) applies.

3. Supposing now that ω^ζ is contributed by some 1β value appearing in the sign sequence of $(\omega^{(ab)^o_k})$, i.e. there is some $k \in \nu ab$ and some $l \in \phi(ab)^o_k$ such that we have some $\hat{\beta}$ ordinal satisfying

$$\sqrt{}(\omega^{\gamma_l(ab)^o_k+1}\beta_l((ab)^o_k)) = \omega^{\gamma_l(ab)^o_l+1}\sqrt{\beta_l(ab)^o_k} = \omega^{\gamma_l(ab)^o_k+1+\hat{\beta}} = \omega^\zeta.$$

[1]This follows immediately from the definition of $\sqrt{}$, since $\sqrt{}$ outputs the first term of the Cantor normal form of each ordinal, so if $\alpha \leq \beta$, then necessarily $\sqrt{\alpha} = \omega^{\alpha_0} \leq \omega^{\beta_0} = \sqrt{\beta}$.

Specifically, $\hat{\beta}$ is such that

$$\gamma_l(ab)^o_k \oplus 1 \oplus \hat{\beta} = \gamma_l(ab)_k \oplus 1 \oplus \hat{\beta} = \gamma_l(a_i + b_j) \oplus 1 \oplus \hat{\beta} = \zeta.$$

Then by Theorem 46, we have

$$\gamma_l(a_i) + \gamma_l(b_j) \oplus 1 \oplus \hat{\beta} \geq \zeta,$$

and since $\sqrt{\iota}(a_i+b_j) \leq \max\{\sqrt{\iota}a_i, \sqrt{\iota}b_j\}$, $\max\{\sqrt{\iota}\omega^{a_i}\} \leq \sqrt{\iota}a$ and similarly $\max\{\sqrt{\iota}\omega^{b_j}\} \leq \sqrt{\iota}b$, we find

$$\omega^{\zeta} \leq \sqrt{\iota}\omega(a_i + b_j) \leq \max\{\sqrt{\iota}a, \sqrt{\iota}b\}$$

from which

$$\omega^{\zeta} \leq \sqrt{\iota}a\sqrt{\iota}b$$

follows.

4. If neither (1), nor (2), nor (3), then we have $\omega^{\zeta} = \sqrt{}(\omega^{(ab)^+_k}\iota(t_k))$, so unless $t_k \in \mathbb{R}\backslash\mathbb{D}$, we have either $\omega^{\zeta} = \omega^{(ab)^+_k} \leq \omega^{a^+_i}\omega^{b^+_j}$, from which we can conclude that $\omega^{\zeta} \leq \sqrt{\iota}a\sqrt{\iota}b$. If $t_k \in \mathbb{R}$, then we have $\omega^{\zeta} = \omega^{(ab)^+_k+1}$, with the inequality following as before provided we note that $t_k = r_i s_j$ where at least r_i or s_j is a non-dyadic rational (since the product of two dyadic rationals is a dyadic rational). From this fact, we find that

$$\omega^{\zeta} \leq \omega^{a^+_i}\omega^{b^+_j}\omega \leq \sqrt{\iota}(\omega^{a_i}r_i)\sqrt{\iota}(\omega^{b_j}s_j) \leq \sqrt{\iota}a\sqrt{\iota}b.$$

Thus we find that $\sqrt{\iota}$ forms a pseudo-absolute value on the surreal numbers. $\square$

Finally, without loss of generality, we'll implicitly have precomposed with ι when we use $\sqrt{}$ in subsequent Chapters, instead of $\sqrt{\iota}$.

CHAPTER 5

VEBLEN HIERARCHY

The primary goal of this Chapter is to prove that the length of every genetic function is bounded above by some Veblen function. The least such ordinal classifying the Veblen function binding the length of a genetic function $\mathfrak{g}$ is the **Veblen rank** of $\mathfrak{g}$. We then prove that the ring of genetic functions generated by a set of genetic functions also has a Veblen rank, and further that this Veblen rank corresponds to the height of the prime $\mathcal{G}$-closed subtree.

Before proceeding with the above construction, recall that at the start of Chapter 4.2 we discussed three Classes of limit ordinals. It has been known that there is a correspondence between the surreal number trees truncated at these corresponding ordinals and analogous theories to the ones described by the limit ordinals. We summarize this correspondence below:

Gamma $\mathbf{No}(\lambda)$ is an additive subgroup of $\mathbf{No}$ if and only if λ is a Gamma ordinal of the form ω^α for some ordinal α.

Delta $\mathbf{No}(\lambda)$ is a commutative subring of $\mathbf{No}$ if and only if λ is a Delta ordinal of the form ω^γ for some Gamma ordinal γ.

Epsilon $\mathbf{No}(\lambda)$ is a real-closed subfield of $\mathbf{No}$ if and only if λ is an Epsilon number.

Moreover, our motivation for finding an appropriate pseudo-absolute value is precisely to make sure that we accurately track the complexity of the image of a genetic functions whose arguments are of a length below a given Epsilon number. Since Epsilon numbers are also Delta numbers, this allows us to begin by reasoning about polynomials, and inducting from there.

5.1 Veblen Rank

Next, we recall the general fixed point theorems of [1] (Theorems 9.4 and 9.4a), summarized below as the following theorem:

Theorem 50. *Suppose* $f\colon \mathrm{No} \to \mathrm{No}$ *satisfies the following properties:*

1. *For all* $a \in \mathrm{No}$, $f(a)$ *is a power of* ω*;*
2. $a < b \Rightarrow f(a) < f(b)$*;*
3. *There are two fixed sets* C *and* D *such that whenever* $a = G|H$*, such that* G *contains no maximum and* H *contains no minimum, then* $f(a) = (C \cup f(G))|(D \cup f(H))$.

Then the function g *defined by*

$$g(b) := \left\{f^{(n)}(C), f^{(n)}(2g(b^L))\right\} | \left\{f^{(n)}(D), f^{(n)}(\frac{1}{2}g(b^R))\right\}$$

is onto the set of all fixed points of f *and satisfies the above hypotheses with respect to the sets* $f^{(n)}(C)$ *and* $f^{(n)}(D)$*, where* $f^{(n)}$ *denotes the* n^{th} *iterate of* f*.*

Furthermore, there is a On*-length family of functions* f_α *satisfying all three conditions, such that* $f_0 = f$ *and for* $\alpha > 0$*,* f_α *is onto the set of all common fixed points of* f_β *for* $\beta \in \alpha$ *and satisfies condition (iii) with respect to the sets* $h(C)$ *and* $h(D)$ *where* h *runs through all finite compositions of* f_β *for* $\beta \in \alpha$*.*

Our goal is to find a uniform way to bound the growth of complexity of a genetic function by some strictly increasing function. The motivation for this is that such a function will give us a coarse way of finding initial subtrees of No which are closed under the application of a genetic function. One way we may systematically study the growth would be with the Veblen hierarchy (first described in [34]):

Definition 61. *A normal ordinal valued function* φ_0 *is any continuous (with respect to the order topology) strictly increasing ordinal valued function. Given a normal function* φ_0*, the* ***Veblen functions*** *with respect to* φ_0 *are the sequence of functions* $\langle \varphi_\alpha \colon \alpha \in \mathrm{On} \rangle$ *such that each* φ_α *enumerates the common fixed points of* φ_β *for every* $\beta \in \alpha$*. The* ***Veblen hierarchy*** *is the class of functions* $\langle \varphi_\alpha \colon \alpha \in \mathrm{On} \rangle$ *generated by* $\varphi_0(x) = \omega^x$*.*
Finally, we have the following ordering on the Veblen hierarchy:

$$\varphi_\alpha(\beta) < \varphi_\gamma(\delta) \iff (\alpha = \gamma \wedge \beta < \delta) \vee (\alpha < \gamma \wedge \beta < \varphi_\gamma(\delta)) \vee (\alpha > \gamma \wedge \varphi_\alpha(\beta) < \delta))$$

Because ω is a genetic function, it is immediate that every Veblen function is a genetic function following the construction found in Theorem 50 when we take $C = \{0\}$ and $D = \emptyset$. In fact, we could show that the construction of g in Theorem 50 given $\varphi_0(x) = \omega(x)$ is equicofinal with the construction of $\epsilon(x)$.
Our primary motivation here is to identify for every $g \in \mathcal{G}$, the least α such that for all $\gamma \in \mathrm{On}$, if $x \in \mathrm{No}(\gamma)$ then $g(x) \in \mathrm{No}(\varphi_\alpha(\gamma))$. We inductively define the notion of partial Veblen rank as follows:

Definition 62. *Fix* $\gamma \in \mathrm{On}$*. The* ***partial Veblen rank*** $\mathsf{VR}(g,\gamma)$ *is defined as follows:*

1. $\mathsf{VR}(g,\gamma) \geq 0$*;*

2. $\mathsf{VR}(g,\gamma) \geq \lambda$ *for limit ordinals* λ *if and only if* $\mathsf{VR}(g,\gamma) \geq \beta$ *for all* $\beta \in \lambda$*;*

3. $\mathsf{VR}(g,\gamma) \geq \alpha + 1$ *if and only if there is an* $x \in \mathrm{No}(\epsilon_\gamma)$ *such that* $\sqrt{g(x)} \geq \varphi_{\alpha+1}(\gamma)$*.*

We say $\mathsf{VR}(g,\gamma) = \alpha$ *whenever* $\mathsf{VR}(g,\gamma) \geq \alpha$ *and* $\mathsf{VR}(g,\gamma) \not\geq \alpha + 1$*, i.e.* α *is the least ordinal such that for all* $x \in \mathrm{No}(\varphi_1(\gamma))$*,* $g(x) \in \mathrm{No}(\varphi_{\alpha+1}(\gamma))$*.*

We then define the ***Veblen rank*** *of* g *by* $\mathsf{VR}(g) := \bigcup_{\gamma \in \mathrm{On}} \mathsf{VR}(g,\gamma)$*.*

We can extend this definition to $g : \mathrm{No}^n \to \mathrm{No}$ *by noting that* $\iota(\bar{x})$ *is the Hessenberg sum of the lengths of the components, so we can interpret* $\mathrm{No}^n(\epsilon_\gamma)$ *as the initial subset of* No^n *consisting of n-tuples of branches whose Hessenberg sum is less than* ϵ_γ*.*

Whenever $\mathsf{VR}(g) \geq \alpha$ *for all* $\alpha \in \mathrm{On}$*, rather than denote this by saying the rank is* ∞*, we indicate this by saying the rank is* On*.*

Proposition 13. *For any* $g : \mathrm{No} \to \mathrm{No}$*, let* $\mathsf{VR}_g : \mathrm{On} \to \mathrm{On}$ *be given by* $\gamma \mapsto \mathsf{VR}(g,\gamma)$*. Then* VR_g *is a proper, ordinal valued function.*

Proof. Since every $g(\mathrm{No}(\epsilon_\gamma)) \subsetneq \mathrm{No}$ is a proper subset of No, VR_g will send γ to some ordinal. Specifically, VR_g identifies the least α such that for each $x \in \mathrm{No}(\epsilon_\gamma)$, $\sqrt{g(x)} < \varphi_{\alpha+1}(\gamma)$, so $g(\mathrm{No}(\epsilon_\gamma)) \subseteq \mathrm{No}(\varphi_{\alpha+1}(\gamma))$ (while it may be the case that $g(\mathrm{No}(\epsilon_\gamma)) \subset \mathrm{No}(\varphi_\alpha(\gamma))$ if α is a limit ordinal). Since VR_g is defined with respect to the unique minimum ordinal respecting the set containment of the image of the ground field, and ON is well-founded, VR_g will be a well-defined function. □

Proposition 14. *The sequence defined by* VR_g *either has an* ON*-length subsequence that is constant, or there is a strictly monotonically increasing* ON*-length subsequence.*

Proof. This follows directly from ON being well-founded, so VR_g"ON will have a least element, so there cannot be a monotonically decreasing ON-length subsequence. If there is an ON-length constant subsequence, i.e. for some $\alpha \in$ ON, $\mathsf{VR}_g^{-1}(\alpha)$ is a proper class, then we're done. This is always guaranteed to happen whenever VR_g"ON is a proper set, as there are $\alpha \in \beta \in$ ON such that VR_g"ON $\subseteq [\alpha, \beta] \subseteq [0, \beta]$.

Towards a contradiction, if each fibre at $\gamma \in [\alpha, \beta]$ forms a proper set, then we have ON $= \bigcup_{\gamma \leq \beta} \mathsf{VR}_g^{-1}(\gamma)$, which is absurd, because then ON would also be a proper set, as the set-union of sets. Thus at least one fibre must be a proper class.

So now without loss of generality, suppose for each $\alpha \in$ ON, $\mathsf{VR}_g^{-1}(\alpha)$ is a proper set. Since the ordinals are well founded, we can form our strictly monotonic increasing ON-length subsequence by taking the least element of the following sets: Let β_0 be the minimum of element of VR_g"ON, set α_0 to be the least element of $\mathsf{VR}_g^{-1}(\beta_0)$, and set $B_0 = \alpha_0 \cup \{\beta_0\}$. Then inductively form:

$$\beta_{i+1} = \min(\mathsf{VR}_g\text{"}(\mathrm{ON} \backslash (\alpha_i + 1))) \backslash B_i$$

$$\alpha_{i+1} = \min(\mathsf{VR}_g^{-1}(\beta_{i+1}) \backslash (\alpha_i + 1))$$

$$B_{i+1} = B_i \cup \{\beta_{i+1}\}$$

and for limit ordinal stages λ,

$$\beta_\lambda = \min(\mathsf{VR}_g\text{"}(\mathrm{On}\setminus\bigcup_{i\in\lambda}(\alpha_i+1)))\setminus\bigcup_{i\in\lambda}B_i$$

$$\alpha_\lambda = \min \mathsf{VR}_g^{-1}(\beta_\lambda)\setminus\left(\bigcup_{i\in\lambda}\alpha_i+1\right)$$

$$B_\lambda = \bigcup_{i\in\lambda}B_i \cup\{\beta_\lambda\}$$

Since minimum elements are guaranteed to exist by well-foundedness, each step of this construction will give us a pair $\langle\alpha_i,\beta_i\rangle$ such that $(\langle\alpha_i,\beta_i\rangle)_{i\in\mathrm{On}}$ is a sequence of pairs satisfying

1. $\mathsf{VR}_g(\alpha_i)=\beta_i$;
2. for all $j<i\in\mathrm{On}$, $\alpha_j<\alpha_i$ and $\beta_i<\beta_j$.

This sequence is an On-length subsequence of the ordinals that is strictly monotonically increasing. □

A consequence of Proposition 14 is the following corollary

Corollary 10. $\mathsf{VR}(g)\in\mathrm{On}$ *if and only if* $\mathsf{VR}_g\text{"}\mathrm{On}$ *is a proper set.*

Proof. In the converse direction, this follows from there being a minimum $\beta\in\mathrm{On}$ such that $\mathsf{VR}_g\text{"}\mathrm{On}\subseteq[0,\beta]$. If β is a limit ordinal and $\mathsf{VR}_g(\alpha)<\beta$ for all $\alpha\in\mathrm{On}$, then $\mathsf{VR}(g)=\bigcup_{\alpha\in\mathrm{On}}\mathsf{VR}(g,\alpha)=\beta$ by the minimality of β. On the other hand, since $\mathsf{VR}_g(\alpha)\le\beta$ for all $\alpha\in\mathrm{On}$, with equality for at least one $\alpha\in\mathrm{On}$, we have $\mathsf{VR}_g(\alpha)=\beta$ following immediately.

To prove the forward direction, we prove the contrapositive. That is, suppose $\mathsf{VR}_g\text{"}\mathrm{On}$ is a proper class. Then it is unbounded, and by Proposition 14 there is an ON length strictly monotonically increasing subsequence, by which $\mathsf{VR}(g) = \bigcup_{\alpha \in \mathrm{On}} \mathsf{VR}(g, \alpha) = \mathrm{On}$, i.e. $\mathsf{VR}(g) \notin \mathrm{On}$. □

Proposition 15. *The following functions are all of Veblen rank 0:*

1. *Identity*
2. *Addition;*
3. *Negation;*
4. *Multiplication;*
5. exp*;*
6. ω.

Proof. Supposing that the identity function had Veblen rank ≥ 1, then there would exist some $x \in \mathrm{No}$ such that $\sqrt{x} > \omega(\iota x)$. However, this is absurd since

$$\sqrt{x} \leq \iota x \leq \omega(\iota x)$$

for all $x \in \mathrm{No}$. Thus the identity function is not at least rank 1, whence the identity function must be rank 0.

Functions (2)-(5) are rank 0 follows from results in [2, 5] showing that trees of height gamma ordinals are closed under addition and negation, and trees of height delta ordinals are closed under multiplication, and finally $\mathrm{No}(\epsilon_\alpha) \models \mathbb{R}_{\exp}$.

Similarly, since

$$\sqrt{\omega^x} \leq \iota\omega^x \leq \omega^{\iota x} < \omega^{\epsilon_\alpha} = \epsilon_\alpha,$$

for all $x \in \mathrm{No}(\epsilon_\alpha)$, it follows that $\mathsf{VR}(\omega, \alpha) = 0$ for all α, whence $\mathsf{VR}(\omega) = 0$. □

Following up with this, the Veblen functions correspond to the Veblen ranks.

Theorem 51. *For every $\alpha \in \mathrm{On}$, the Veblen function φ_α has Veblen rank α. Furthermore,* $\beth(\varphi_\alpha) = \begin{cases} \alpha + 1 & \alpha \in \omega \\ \alpha & \alpha \geq \omega \end{cases}$

Proof. First, for all $x \in \mathrm{No}(\epsilon_\gamma)$, we have $\sqrt{\varphi_\alpha(x)} \leq \iota\varphi_\alpha(x) \leq \varphi_\alpha(\iota x) < \varphi_\alpha(\epsilon_\gamma)$. So we may as well work with the cases where $x \in \mathrm{No}(\epsilon_\gamma)$ is an ordinal.

Then, for all $\alpha \in \mathrm{On}$, $\mathsf{VR}(\varphi_\alpha, 0) = \alpha$, since for all $\gamma \in \epsilon_0$ and all $\beta \in \alpha$, we have $\varphi_\beta(\gamma) \leq \varphi_\alpha(\gamma) < \varphi_{\alpha+1}(0)$, whence $\mathsf{VR}(\varphi_\alpha) \geq \alpha$.

However, in general $\mathsf{VR}(\varphi_\alpha, \gamma) = \alpha$ since for all $\delta \in \epsilon_\gamma$, and $\beta \in \alpha$, we have $\varphi_\beta(\delta) \leq \varphi_\alpha(\delta) < \varphi_{\alpha+1}(\gamma)$, by the inequality condition on the Veblen hierarchy, namely, $\delta < \epsilon_\gamma \leq \varphi_{\alpha+1}(\gamma)$.

The "furthermore" follows by induction. Starting with $\alpha_0 = \omega$, since

$$\omega(x) := \left\{0, \omega(x^L)n\right\} \mid \left\{\omega(x^R)2^{-n}\right\},$$

it is immediate that $\beth(\omega) = \beth(\varphi_0) = 1$ by definition of generation. Moreover, because S_1 is closed under addition, multiplication, and composition, every $\beth(\omega^{(n)}) = 1$. From this, by definition of $\beth$, $\beth(\varphi_1) = 2$ since it depends on a set of generation 1 functions.

By induction, suppose this is true up for all $n \in \alpha \in \omega$. Then φ_α will be defined with respect to $\varphi_0, \ldots, \varphi_n$, from which it follows $\daleth\varphi_\alpha = \alpha + 1$.

Now suppose this is true for all $n \in \omega$. It follows that $\daleth(\varphi_\omega) = \omega$ by our induction hypothesis, since the set of ancestors of φ_ω is generated by $\{\varphi_i : i \in \omega\}$. Finally, supposing this is true up to some infinite α, then again, by our induction hypothesis, the set of ancestors of φ_α is generated by $\{\varphi_\beta : \beta \in \alpha\}$, from which $\daleth(\varphi_\alpha) = \alpha$. □

Proposition 16. *Every constant function* $c : \mathrm{No} \to \mathrm{No}$ *has Veblen rank* $\alpha = \mathsf{VR}(c, 0)$ *and* VR_c *is monotonically decreasing, eventually zero function.*

Proof. Since c is a surreal number, c can be understood as the function $c : \gamma \to 2$, i.e. there is a minimal γ such that $c \in \mathrm{No}(\gamma + 1)$ and $c \notin \mathrm{No}(\beta)$, for all $\beta \leq \gamma$. Furthermore, there is some a minimal δ such that $\gamma \in \epsilon_\delta$ but not in ϵ_η for $\eta \in \delta$. It follows immediately that for all $\chi \geq \delta$, $\mathsf{VR}_c(\delta) = 0$, since $\epsilon_\delta \leq \epsilon_\chi$. From this we conclude that VR_c is non-zero on the set δ. Specifically, since $\gamma \notin \epsilon_\eta$ for $\eta \in \delta$, we have $c \notin \mathrm{No}(\epsilon_\eta)$, and so we must have $\mathsf{VR}_c(\eta) > 0$. Further, since $\varphi_{\alpha+1}(0)$ will be a fixed-point of ϵ_x, it must be the case that $\epsilon_\delta \leq \varphi_{\alpha+1}(0)$. If $\alpha = 0$, then $\delta = 0$, and otherwise, we have $\delta < \varphi_{\alpha+1}(0)$, or $\delta = \varphi_{\alpha+1}(0)$.

We see that VR_c takes maximum value $\alpha = \mathsf{VR}(c, 0)$ because as a constant map, c necessarily sends $\mathrm{No}(\epsilon_0)$ to some specific $c \in \mathrm{No}(\varphi_{\alpha+1}(0))$. If there is some $\xi \ni 0$ such that $\mathsf{VR}_c(\xi) = \zeta > \alpha$, then because $\zeta > \alpha$ and $\xi > 0$, we must have $\varphi_{\alpha+1}(0) < \varphi_{\zeta+1}(\xi)$, which is absurd, as VR_c sends ordinal ξ to the least ordinal α' such that $\iota c < \varphi_{\alpha'+1}(\xi)$

□

Our immediate goal is to show that given $\mathcal{G} = \{g_i : i \in I\}$, where I is an index set, and each g_i such that each $\mathsf{VR}(g_i) = \alpha_i$, then the ring of functions generated by $\mathcal{G}$ will have a corresponding Veblen rank at most $\sup_I\{\alpha_i\}$. Towards that end, we verify the following Lemmas which will be needed for this result.

Lemma 10. *For all surreal-valued functions* f, g *of the same arity,* $\mathsf{VR}(f+g) \leq \max\{\mathsf{VR}(f), \mathsf{VR}(g)\}$.

Proof. This follows by $\surd$ having the ultrametric property and Theorem 51. Precisely, suppose that $\mathsf{VR}(f) = \alpha$ and $\mathsf{VR}(g) = \beta$, then

$$\surd(f(\bar{x}) + g(\bar{x})) \leq \max\{\surd f(\bar{x}), \surd g(\bar{x})\} \leq \max\{\varphi_\alpha(\iota(\bar{x})), \varphi_\beta(\iota(\bar{x}))\}$$

and then by Theorem 51, $f + g$ will have Veblen rank at most $\max\{\alpha, \beta\}$.

□

In some sense, for all non-zero ordinals α, each φ_α is the simplest genetic function of Veblen rank α given that we arrive at each φ_α by iterates of the previous φ_β functions composed with φ_α on the predecessors of the argument, and the operations above. This result is summarized best by the following theorems:

Lemma 11. *For all surreal-valued* f, g *of the same arity, we have*

$$\mathsf{VR}(fg) \leq \max\{\mathsf{VR}(f), \mathsf{VR}(g)\}.$$

Proof. First, since $\surd$ is a pseudo-absolute value,

$$\surd((fg)(x)) \leq \surd(f(x))\surd(g(x)) \leq \iota(f(x))\iota(g(x)).$$

Additionally, since every fixed point of ω is multiplicatively indecomposable, whenever $\iota x \in \epsilon_\gamma$, we have $\iota(f(x)) \in \varphi_{\alpha+1}(\gamma)$, and $\iota(g(x)) \in \varphi_{\beta+1}(\gamma)$, so the product $\iota(f(x))\iota(g(x)) \in \varphi_{\max\{\mathsf{VR}(f),\mathsf{VR}(g)\}+1}(\gamma)$. □

Proposition 17. *If* $p \in \mathrm{No}[\bar{x}, \mathcal{G}(\bar{y})]$*, where* $\mathcal{G}(\bar{y})$ *is a set of functions* g_i *of finite arity, by a set* I*, such that for each* $i \in I$*,* $\mathsf{VR}(g_i) = \alpha_i$*, and the constants of* p *are* $c_0, \ldots, c_n$ *such that* $\mathsf{VR}(c_j) = \beta_j$*, then* $\mathsf{VR}(p) \leq \max\{\alpha_i, \beta_j \colon i \in I, j \in [n]\}$.

Proof. First, we note that for each monomial term $c_j g_{i_1} \cdot g_{i_n} x^J$ of p, where J is a multi-index for $\bar{x}$, we have by Lemma 12 that $\mathsf{VR}(c_j g_{i_1}(\bar{x}) \cdot g_{i_n}(\bar{x}) x^J) \leq \max\{\beta_j, \alpha_{i_1}, \ldots, \alpha_{i_n}\}$. In particular, $\mathsf{VR}(\bar{x}^J) = 0$.

Then by Lemma 10, $\mathsf{VR}(p) \leq \max\{\mathsf{VR}(c_j g_{i_1} \cdot g_{i_n} x^J)\}$, from which we have

$$\mathsf{VR}(p) \leq \max\{\alpha_i, \beta_j\}.$$

□

Lemma 12. *For all* f, g*,* $\mathsf{VR}(f \circ g) \leq \max\{\mathsf{VR}(f), \mathsf{VR}(g)\}$.

Proof. Without loss of generality, let $\mathsf{VR}(f) = \alpha$ and $\mathsf{VR}(g) = \beta$, since if either is has Veblen rank On, there is an On-length subsequence of $\mathsf{VR}_{f \circ g}$ that is strictly monotonically increasing.

Fixing $\gamma \in \textsc{On}$, and supposing that $x \in \varphi_{\beta+1}(\gamma)$. If $\alpha > \beta$, then $\varphi_{\beta+1}(\gamma) < \varphi_{\alpha+1}(\gamma)$, since $\beta + 1 \leq \alpha$, so $\varphi_{\alpha}(x) < \varphi_{\alpha+1}(\gamma)$. If $\alpha = \beta$, this is also immediate. Finally, if $\alpha < \beta$, since $\varphi_{\beta+1}(\gamma)$ will enumerate fixed points of $\varphi_{\alpha+1}$, $\varphi_{\alpha+1}(\varphi_{\beta+1}(\gamma)) = \varphi_{\beta+1}(\gamma)$.

Thus $\mathsf{VR}(f \circ g) \leq \max\{\mathsf{VR}(f), \mathsf{VR}(g)\}$. □

Lemma 13. *Let* S *be a set of genetic functions. Then, for any term* t *formed formed using* $\{0, 1, +, -, \times\} \cup S$*, we find that*

$$\mathsf{VR}(t^n) \leq \mathsf{VR}(t) \leq \sup\{\mathsf{VR}(g) \colon g \in S\}.$$

Proof. We induct on the complexity of terms. Immediately, $\mathsf{VR}(v_i) = 0$ for all variables v_i, along with $\mathsf{VR}(+) = \mathsf{VR}(\times) = 0$. Further, for each $g_i \in S$, we have $\mathsf{VR}(g_i) = \alpha_i$ by our hypothesis. The result for both equalities follows by combining Theorems 10, 11, 12, and 17. Finally, the result for partial Veblen rank follows by a routine induction argument on γ, and application of the results cited in the previous sentence providing an upper bound on partial rank. □

Collecting Lemmas 10, 11, 12, and 13, we have the following theorem:

Theorem 52. *Given* $\mathcal{G}$*, a set of genetic functions indexed by set* I*, and such that* $\mathsf{VR}(g_i) = \alpha_i$*. Then* $\mathsf{VR}(\mathcal{G}^*) = \sup_I \alpha_i$.

The following theorem will be used to establish that every genetic function has a Veblen rank below $\textsc{On}$.

Theorem 53. *Suppose that* f *is a genetic function whose Left and Right options sets has order type* τ*, i.e.* $\mathsf{o.t.}(L_f \cup R_f) = \tau$*, and* $L_f \cup R_f$ *consists of genetic functions* g_i *indexed by some set* I*, such that for each* $i \in I$*,* $\mathsf{VR}(g_i) = \alpha_i$*. Set* $\alpha = \sup_I \alpha_i$*, and* $\mu = \max\{\tau, \alpha\}$*. Then* $\mathsf{VR}(f) \leq \mu + 1$*. Further,* $\mathsf{VR}(f) = \mu + 1$ *if and only if for at least one* $\gamma \in \mathrm{On}$*, there is some* $x_\gamma \in \mathrm{No}(\epsilon_\gamma)$ *for which there is an infinite enumeration* K *of terms in* $L_f \cup R_f$*, such that for* $k, k' \in K$*,* $\varphi_k(\gamma) \leq \sqrt{t_k(x_\gamma)} \leq \sqrt{t_{k'}(x_\gamma)}$ *when* $k < k'$ *and such the sequence* $(\sqrt{t_k(x_\gamma)})$ *is cofinal with* $\varphi_{\mu+1}(\gamma)$*.*

Proof. First, we may as well suppose that the result already holds for every genetic function symbol appearing in the Option sets for f.

Next, we recall that every term $t \in L_f \cup R_f$ can be decomposed into a sum-product of linear forms $\sum_i \prod_j t_{ij}$, where each $t_{ij} = c_1 + c_2 + h(c_3 x + c_4)$ where for some set of genetic function symbols S closed under composition, we have $c_1, c_2, c_3, c_4 \in S[\{g(v), g(w) \colon g \in S \cup \{f\}\}]$ and $h \in S$.

Here, we without loss of generality we may take $S = \{g_i \colon i \in I\}$, and we assume that we have substituted x^L for v and x^R for w in the terms.

For terms t where the function symbol for f does not appear, by Lemmas 10, 11, and 13, we have $\mathsf{VR}(t) \leq \max\limits_{i,j}\{\mathsf{VR}(t_{ij})\}$, and for some t_{ij} with maximum Veblen rank, this in turn is bounded above by

$$\max\{\mathsf{VR}(c_1), \mathsf{VR}(c_2), \mathsf{VR}(c_3), \mathsf{VR}(c_4), \mathsf{VR}(h)\}.$$

We need to induct on the complexity of the argument x across all terms in order to account for the terms where $f(x^L), f(x^R)$ appear in order to extend the application of Lemmas 10, 11, and 13.

When $x = 0$, the only terms that are applied are the constants in the option sets, and the terms defined without the f symbol (e.g. terms composed of $g_i(0)$). In this case, $\sqrt{f} \leq \sup \sqrt{g_i(0) + 1}$, which in turn suggests $\sqrt{f(0)}$ is at most $\varphi_{\alpha+1}(0)$, as the observation above will still hold on these terms previously made holds.

So now suppose that for all $x \in \text{No}$ such that for all $\iota x < \beta \in \epsilon_\gamma$, we have $\sqrt{f(x)} < \varphi_{\mu+2}(\gamma)$. Then for any x such that $\iota x = \beta$, by the induction hypothesis, $\sqrt{f(x')} \leq \varphi_{\mu+1}(\gamma)$ and by our initial hypotheses that $\sqrt{g_i(x)} < \varphi_\mu(\gamma)$ for each $i \in I$, from which we can apply Lemma 12 supposing that below ϵ_γ. For the terms with f evaluated on the predecessors of x, the Veblen rank is at most $\mu + 1$. In this case, since we have $\tau \leq \mu$ many terms $t_k(x)$, such that $\sqrt{t_k(x)} < \varphi_{\mu+2}(\gamma)$, without loss of generality we may consider the most extreme case, where $K = \tau$, and

$$\varphi_k(\gamma) \leq \sqrt{t_k(x)} < \sqrt{t_{k'}(x)}.$$

By the induction hypothesis, we may further suppose that $x \sqsupset x_\gamma$, so that $\varphi_{\mu+1}(\gamma) \leq \sqrt{f(x_\gamma)} < \varphi_{\mu+2}(\gamma)$. Since each term is a polynomial that we can break down into a sum-product form to which we can partially apply Lemma 17 (we can does this because $VR(g_i) \leq \mu + 1$ and the terms are evaluated on the image of f applied to predecessors of x), we can conclude $\sqrt{t_k(x)} < \varphi_{\mu+2}(\gamma)$ for all $k \in \tau$. Further, no sequence of polynomial terms written with constants in $\text{No}(\varphi_{\mu+2})$ and functions g_i with Veblen rank $\leq \mu$ will be cofinal with $\varphi_{\mu+2}(\gamma)$.

To see this, for ease of computation and without loss of generality, suppose that $\iota f(x_\gamma) = \varphi_{\mu+1}(\gamma)$ (though $f(x_\gamma)$ can be of length below $\varphi_{\mu+2}(\gamma)$, we may without loss of generality suppose that it is a Veblen ordinal). Then because for each $k \in \tau$, we have $\sqrt{t_k} \leq \max_i \prod_j \sqrt{t_{k_{ij}}}$, where $t_{k_{ij}} = \prod_{j \leq n_i}(c_{1_{ij}} + c_{2_{ij}} g_{ij}(c_{3_{ij}} x + c_{4_{ij}})$, and for each c term, we have $\sqrt{c} \leq \sqrt{f(x_\gamma)} = f(x_\gamma) = \varphi_{\mu+1}(\gamma)$ by our simplifying assumption, and $VR(g_{ij}) \leq \alpha \leq \mu < \mu + 1$ implies that $\sqrt{(g_{ij}(c_3 x + c_4))} < \varphi_{\alpha+1}(\varphi_{\mu+2}(\gamma)) = \varphi_{\mu+2}(\gamma)$, we have

$$\sqrt{t_k(x)} \leq \varphi_{\mu+1}(x)^{n_i}.$$

Similarly, since $\iota x_\gamma < \varphi_{\mu+2}(\gamma)$, we can bound $\iota x_\gamma < \varphi_{\mu+1}(\gamma + \xi)$ for some ordinal ξ such that $\gamma + \xi < \varphi_{\mu+2}(\gamma)$, from which we would have

$$\sqrt{t_k(x)} < \varphi_{\mu+1}(\gamma + \xi)^{n_i} < \varphi_{\mu+2}(\gamma).$$

Since simplicity will only apply whenever we have a fixed x and its predecessors, for all $k \in \tau$, we have

$$\sqrt{t_k(x)} < \varphi_{\mu+1}(\gamma + \xi)^{\omega} < \varphi_{\mu+2}(\gamma),$$

in which case by simplicity, $\sqrt{f(x)} < \varphi_{\mu+2}(\gamma)$.

From this we conclude that $VR(f, \gamma) \leq \mu + 1$. By setting $\beta = \epsilon_\gamma \in \epsilon_{\gamma+1}$, we can repeat the same argument, so by transfinite induction, we find that $VR(f) \leq \mu + 1$.

Now to see why the conditions for equality hold, we will first prove the contrapositive of the forward direction. Specifically, if we suppose that for all $\gamma \in \mathrm{On}$ for all $x_\gamma \in \mathrm{No}(\epsilon_\gamma)$, for all subsets K of $L_f \cup R_f$ enumerating terms, $(\sqrt{(t_k(x_\gamma))})$ is cofinal to an element in $\varphi_{\mu+1}(\gamma)$, then $\mathsf{VR}(f, \gamma) \leq \mu < \mu + 1$.

On the other hand, in the reverse direction, the equality will hold as simplicity will entail that $\sqrt{f(x_\gamma)} \geq \varphi_{\mu+1}(\gamma)$, and the argument above will show that $\sqrt{f(x)} < \varphi_{\mu+2}(\gamma)$ in general. □

Since every genetic function is defined with respect to *sets* of previously defined genetic functions, we have:

Corollary 11. *Every genetic function* g *has a bounded Veblen rank, i.e. if* $g \in \mathcal{G}$*, then* $\mathsf{VR}(g) \in \mathrm{On}$*. Further, for all sets* $\mathcal{G} \subsetneq \mathcal{G}$*,* $\mathsf{VR}(\mathcal{G}) \in \mathrm{On}$*.*

Proof. This can be demonstrated by inducting on the complexity of genetic functions, and application of Theorem 53. □

As a further consequence of Lemma 13 and Corollary 11, we can extend our notion of Veblen rank to formula and theories, as follows:

Definition 63. *Let* $\mathcal{G}$ *be a set of genetic functions, and* $\mathcal{L} = \mathcal{L}_{\mathsf{oring}} \cup \mathcal{G}$*. Regarding every* $\mathcal{L}$*-term as a surreal valued function, we define the Veblen rank any (definable)* $\mathcal{L}$*-formula* ϕ *to be the maximum Veblen rank of the terms* $t_i(\bar{x})$ *appearing* ϕ*, and similarly we define the partial Veblen rank for* ϕ *as* $\mathsf{VR}(\phi, \gamma) = \max \mathsf{VR}(t_i, \gamma)$ *such that* $\iota\bar{x} \leq \gamma$*, with strict inequality whenever* γ *is a fixed point of* ω*. Finally, for all* $\mathcal{L}$ *theories* T *such that* $\mathrm{No} \models T$*, we set the Veblen rank of* T *to be the supremum of the Veblen ranks of all sentences* $\psi \in T$*. Similarly, we define the partial Veblen rank to be* $\mathsf{VR}(T, \gamma) = \sup\{\mathsf{VR}(\phi, \gamma) \colon \phi \in T\}$*.*

In addition to being to extend our notion of Veblen rank to sets of sentences and formulas in terms of entire genetic functions. Specifically, we can extend our notion of Veblen rank to genetic functions that are inverses to entire genetic functions whose image is a convex proper subclass of No. If the image is a proper convex subclass, there is a simplest element from which we can run an induction argument over, and the proper convex subclass can still be subject to the truncation necessary for studying the complexity of the function.

Lemma 14. *If* $f \in \mathcal{G}$ *is an entire surreal-valued genetic function such that* $B := f''\mathrm{No}$ *is a convex interval and* g *is a recursively definable function satisfying the uniformity property on* B *such that* $g \circ f = 1_{\mathrm{No}}$ *and* $f \circ g = 1_B$*, then* $\mathsf{VR}(f) = \mathsf{VR}(g)$.

Proof. Suppose f is an entire genetic function, B, the class image of f, is a convex interval, and we can recursively define a function g using the game construction between left and right options on the interval B such that g has the uniformity property, so that $g \circ f = 1_{\mathrm{No}}$ and $f \circ g = 1_B$.

If $\mathsf{VR}(f) = \alpha$, then we can describe $f = \bigcup_{\mathrm{On}} f_\gamma$ where

$$f_\gamma : \mathrm{No}(\varphi_\alpha(\gamma)) \to \mathrm{No}(\varphi_\alpha(\gamma))$$

for all $\gamma \in \mathrm{On}$, with $f_\gamma \sqsubset f_\lambda$ for all $\gamma \in \lambda$.[1]

Since we have for all $a \in \mathrm{No}(\varphi_\alpha(\gamma))$ that $\iota a < \varphi_\alpha(\gamma)$ and $0 \leq \sqrt{(f(a))} < \varphi_\alpha(\gamma)$ in I, and the image of g is $\mathrm{No}(\varphi_\alpha(\gamma))$, it follows that $\mathsf{VR}(g) = \alpha$.

Let $B_\gamma = B \cap \mathrm{No}(\varphi_\alpha(\gamma))$. If $\mathsf{VR}(g) > \alpha$, then we immediately derive a contradiction, because then there will be some $x \in B_\gamma$ such that $\sqrt{g}(x) \geq \varphi_\alpha(\gamma)$. But since g is an inverse map, this means that there is a $y \in \mathrm{No}(\varphi_\alpha(\gamma)$ such that $y = g(x)$ and $\iota y \geq \varphi_\alpha(\gamma)$, which is absurd. If $\mathsf{VR}(g) < \alpha$, then we similarly derive a contradiction, as $\mathsf{VR}(g) = \beta < \alpha$ with $C_\gamma = B \cap \mathrm{No}(\varphi_\beta(\gamma))$ and $g_\gamma : C_\gamma \to \mathrm{No}(\varphi_\beta(\gamma))$ for all γ entails that for every $x \in C_\gamma$, $\sqrt{g}(x) < \varphi_\beta(\gamma)$. But since $\mathsf{VR}(f) = \alpha > \beta$, there must be some $y \in \mathrm{No}(\varphi_\beta(\gamma))$ such that $\sqrt{f}(y) \geq \varphi_\beta(\gamma)$ for at least one γ. But since g is an inverse of C_γ onto $\mathrm{No}(\varphi_\beta(\gamma))$, it follows that $\sqrt{f}(y) = \sqrt{f}(g(x)) = \sqrt{x} \geq \varphi_\beta(\gamma)$, which is absurd. $\square$

Recall, every convex subclass of No has a simplest element by Conway's simplicity theorem, a recursively definable g will be defined on this simplest element for its base definition. However, it is not necessarily the case that g sends this simplest element to 0, as genetic functions need not be simplicity preserving.

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[1] When understanding the coherence of the union of these maps up to a limit ordinal λ, we should stress that $f_\lambda = \bigcup_\lambda f_\gamma$, with equality on the nose as

$$\bigcup_{\gamma \in \lambda} \mathrm{No}(\varphi_\alpha(\gamma)) = \mathrm{No}(\varphi_\alpha(\lambda)),$$

and so every for every $x \in \mathrm{No}(\varphi_\alpha(\lambda)$, $x \in \mathrm{No}(\varphi_\alpha(\gamma)$ for some γ, and $f_\lambda(x) = f_\gamma(x) \in \mathrm{No}(\varphi_\alpha(\gamma))$.

Example 2. *The first immediate example we have is* log *as the genetically definable inverse of* exp.

Example 3. *The second example we have is the* g *function used to study the structure of* exp. g *is the recursively definable inverse of the* h *function which is used to study the growth of* log, *which we will discuss in Chapter 6.3.*

We now recall the following definition of *tameness* from [4]:

Definition 64. *For* $n \in \mathbb{N}$, *Class-sized linearly ordered Field* $(\mathbb{K}, \leq)$ *with Dedekind completion* $\mathbb{K}^{\mathcal{D}}$, *and* $f : \mathbb{K}^{n+1} \to \mathbb{K}$, *we say* f *is* ***tame*** *if and only if for every* $\bar{d} \in \mathbb{K}^n$, *either* $f(x, \bar{d})$ *is constant, or for every* $\zeta \in \mathbb{K}^{\mathcal{D}} \backslash \mathbb{K}$, *and* $c \in \mathbb{K}$, *there exists* $a, b \in \mathbb{K}$ *such that* $a < \zeta < b$ *and either*

$$\forall x \in (a, \zeta), f(x, \bar{d}) > c \qquad \textit{or}$$

$$\forall x \in (a, \zeta), f(x, \bar{d}) < c.$$

and similarly for (ζ, b).

We say $f : \text{No}^{n+1} \to \text{No}$ *is* ***strongly tame*** *if and only if for all* $a < b \in \text{No}$ *and* $\bar{e} \in \text{No}^n$, $d \in \text{No}$, *either* $f(x, \bar{e})$ *is constant, or there exists* $\zeta_0, \zeta_1, \ldots, \zeta_m \in \text{No}^{\mathcal{D}}$ *such that*

$$a = \zeta_0 < \zeta_1 < \ldots < \zeta_m = b$$

and for $i = 0, \ldots, m - 1$

$$\forall x \in (\zeta_i, \zeta_{i+1}), f(x, \bar{d}) > c \qquad \textit{or}$$

$$\forall x \in (\zeta_0, \zeta_{i+1}), f(x, \bar{d}) < c.$$

Following from the definition of strong tameness and Corollary 11, we prove the following:

Theorem 54. *For every entire genetic* $g \in \mathfrak{G}$*,* g *is strongly tame.*

Proof. Fix $g \in \mathfrak{G}$. By Corollary 11, suppose $\mathsf{VR}(g) = \alpha$. Then for each surreal $a, b, d, \bar{e} \in \mathrm{No}$, let γ be the least ordinal such that $a, b, d, e_i \in A_\gamma$. But then $g(x, \bar{e}) \in A_\gamma$ by the construction of Veblen rank, and so we have that $|g(x, \bar{e})| < \varphi_{\alpha+1}(\gamma) \in \mathrm{No}$. □

We recall another definition from [4]:

Definition 65. *Let* $n \in \mathbb{N}$*, and* $f : \mathbb{K}^{n+1} \to \mathbb{K}$*. We say* f *has the **sup property** if and only if for all* $\bar{d} \in \mathbb{K}^{n+1}$*, for all* $a < b, c \in \mathbb{K}$ *the infimum and supremum of the following classes*

$$\{x \in \mathbb{K} \colon a < x < b \wedge f(x, \bar{d}) \leq c\}$$

$$\{x \in \mathbb{K} \colon a < x < b \wedge f(x, \bar{d}) \geq c\}$$

are in $\mathbb{K} \cup \{\pm\infty\}$*. When* $\mathbb{K} = \mathrm{No}$*, then the infimum and supremum must be in the class* $\mathrm{No} \cup \{\mathrm{On}, \mathrm{Off}\}$*.*

Recall, that following [3], we have banished gaps like ω^{Off}, so that sequence such as $\omega - \alpha$ will converge to 0 in our notion of limit.

This leads to the question whether every genetic function $\mathfrak{g}$ has the sup property, as conceivably there are gaps $\mathfrak{g}$ that can be formed by taking the Dedekind completion with respect to a cut of an interval (a, b) defined with respect to the supremum of a class like

$$\{x \in \mathrm{No}\colon x \in (a, b) \wedge g(x, \bar{d}) \leq c\}.$$

With this in mind, we introduce the following notation:

Notation 5. *Let* $g \in \mathfrak{G}$ *with arity* $|g|$ *and Veblen rank* α. *Let* $\bar{e} = \langle a, b, c, \bar{d} \rangle \in \mathrm{No}^{|g|+2}$, *and let*

$$B[\bar{e}] = \{x \in \mathrm{No}\colon x \in (a, b) \wedge g(x, \bar{d}) \leq c\}.$$

$$C[\bar{e}] = \{x \in \mathrm{No}\colon x \in (a, b) \wedge g(x, \bar{d}) \geq c\}.$$

Recall for all $\gamma \in \mathrm{On}$, *we let* $A_\gamma = \mathrm{No}(\varphi_{\alpha+1}(\gamma))$. *For* $X = B[\bar{e}]$ *or* $C[\bar{e}]$, *and* $\gamma \in \mathrm{On}$, *we set* $X_\gamma[\bar{e}] = X[\bar{e}] \cap A_\gamma$. *Further, set* $Y_\gamma[\bar{e}] = (a, b) \backslash X[\bar{e}]$.

We then let $s_\gamma(X[\bar{e}])$ *and* $m_\gamma(X[\bar{e}])$ *denote the minimal-set theoretic realization of* $\sup X_\gamma[\bar{e}]$ *and* $\inf X_\gamma[\bar{e}]$ *respectively.*[1] *Specifically,* $s_\gamma(X[\bar{e}])$ *is either an element in* X_γ *such that for all* $y \in A_\gamma \cap (a, b)$ *we have* $g(y, \bar{d}) > c$ *if* $X[\bar{e}] = B[\bar{e}]$ *or* $g(y, \bar{d}) < c$ *if otherwise, or is otherwise to be the surreal number* $X_\gamma[\bar{e}]|Y_\gamma[\bar{e}]$, *i.e. the minimal set-theoretic realization of* $(X_\gamma[\bar{e}]|Y_\gamma[\bar{e}])$.

---

[1] *We may refer to this as* local, *in the sense that with respect to the set* A_γ, *we can, set-wise, define a value that is a supremum within a minimal set-sized set-theoretic universe* V_α.

Similarly for $\mathfrak{m}_\gamma(X[\bar{e}]$*, we take care to note that in the case of the minimal cut realization outside of* A_γ*, we must have the infimum be the surreal number defined by* $Y_\gamma([\bar{e}])|X_\gamma([\bar{e}])$*.*

Remark 27. *It is not immediate whether* s_γ *or* $\mathfrak{m}_\gamma$ *are such that* $g(s_\gamma, \bar{d}) \leq c$ *or similarly* $\geq c$*.*

As a direct consequence of our notational definition and minimal set-theoretic realization, we have the following Proposition:

Proposition 18. *With* g*,* γ*,* $\bar{e}$*,* $X[\bar{e}]$*,* X_γ*, and* $s_\gamma(X[\bar{e}])$ *and* $\mathfrak{m}_\gamma(X[\bar{e}])$ *as above, then for all* γ *and for all* $\bar{e}$*,*

1. $s_\gamma(X[\bar{e}]) \in X_{\gamma+1}$*, and*
2. $s_\gamma(X[\bar{e}]) \in X_{\gamma+1} \backslash X_\gamma$ *if and only if* $\iota(s_\gamma(X[\bar{e}])) = \varphi_{\alpha+1}(\gamma)$*.*

and similarly for $\mathfrak{m}_\gamma$*.*

Proof. This follows immediately by the minimal set realization of a cut in A_γ. Either it will be a number of length $\varphi_{\alpha+1}$, as every element in A_γ is of height below $\varphi_{\alpha+1}$, or it will be some element in A_γ. □

Remark 28. *The second item is equivalent to saying that in each set* $A_\gamma \cap (a, b)$*, either the function attains its supremum*

Proposition 19. *For* $g \in \mathfrak{G}$*, if for all* $\bar{e} \in \mathrm{No}^{|g|+2}$ *and all* $X[\bar{e}] \in \{B[\bar{e}], C[\bar{e}]\}$*, there is always a* $\gamma_{\bar{e}}$ *such for all* $\delta \geq \gamma_{\bar{e}}$*,*

$$s_{\gamma_{\bar{e}}}(X[\bar{e}]) = s_\delta(X[\bar{e}])$$

and

$$\mathfrak{m}_{\gamma_{\bar{e}}}(X[\bar{e}]) = \mathfrak{m}_{\delta}(X[\bar{e}]),$$

then $\mathfrak{g}$ *has the sup property.*

Proof. Supposing the hypothesis holds, it is a routine unfolding of the definitions to see that the function has the sup property since the supremum of each class is given by $s_{\gamma_{\bar{e}}}$ and the infimum by $\mathfrak{m}_{\gamma_{\bar{e}}}$. □

We prove the following theorem, and leave the general question open for a future paper:

Theorem 55. *Let* $\mathfrak{g} \in \mathfrak{G}$*. Then* $\mathfrak{g}$ *has the sup property if and only if for all* $\bar{e} = \langle a, b, c, \bar{d} \rangle \in$ $\mathrm{No}^{|\mathfrak{g}|+2}$*, the* On*-length sequences* $\langle s^{\mathfrak{g}}_{\gamma}(X[\bar{e}])\rangle_{\gamma \in \mathrm{On}}$ *and* $\langle \mathfrak{m}^{\mathfrak{g}}_{\gamma}(X[\bar{e}])\rangle_{\gamma \in \mathrm{On}}$ *converge to some* $\sigma_{X[\bar{x}]}, \mu_{X[\bar{e}]} \in$ $\mathrm{No} \cup \{\mathrm{On}, \mathrm{Off}\}$ *respectively for all* $\bar{e}$*, and for* $X[\bar{e}] = B[\bar{e}]$ *or* $C[\bar{e}]$*.*

Proof. This follows by a straightforward unfolding of the definitions, since either we stabilize at some surreal number, or we have an On length strictly increasing (or decreasing sequence) which would approach a gap. If it approaches On or Off, then we have the sup property. If it a approaches a gap in (a, b), then by definition it does not. □

5.2 Extending the notion of Nested Truncation Rank

Berarducci and Mantova [11] introduced their notion of nested truncation rank in part to track the numbers where simplicity was preserved by exp, as exp is not simplicity preserving in general. Tracking this information is necessary in order to establish that the surreal numbers are a transserial Hahn field. As we are interested in establishing that the surreal numbers form what a $\mathcal{G}$-structured Hahn field (see Chapter 7 for details). The following generalizes their notions of nested ranking from exp to arbitrary entire genetic functions. We then put some additional hypotheses on the properties of the genetic functions in $\mathcal{G}$ as well as properties on $\mathcal{G}$ itself to describe the behavior of the corresponding partial ordering that we can put on the surreal numbers.

Definition 66. *Let $g \in \mathcal{G}$. Below we consider finite sums of surreal numbers in standard form, and use the convention that for $x \in \mathrm{No}^\times$ we set sgn$(x) = 1$ if $x > 0$ and sgn$(x) = -1$ otherwise.*

We then inductively define rank a generalized notion of truncation rank $\trianglelefteq^g_n$ on $\mathrm{No}^\times$ over $n \in \omega$ as follows:

1. $x \trianglelefteq^g_0 y$ *if* $x \trianglelefteq y$;
2. $x \trianglelefteq^g_{n+1} y$ *if there are* $a \trianglelefteq^g_n b$ *with* $a, b \in \mathbb{J}^*$, *and* $z, w \in \mathrm{No}$ *and* $r \in \mathbb{R}^\times$ *such that*

$$x = z + \mathit{sgn}(r)g(a)$$

$$y = z + rg(b) + w$$

where both sums are in standard form.

We say $x \trianglelefteq^g y$, or that x is a ***nested g-truncation of y*** *if there is an n such that $x \trianglelefteq^g_n y$.*

$\trianglelefteq^g$ induces a foundation rank, which we define as follows:

For a genetic function $\mathsf{g} \in \mathcal{G}$*, and for all* $x \in \text{No}^{\times}$*, the* ***nested g-truncation rank****,* $\mathsf{NR}_{\mathsf{g}}(x)$ *is defined by*

$$\mathsf{NR}_{\mathsf{g}}(x) := \sup\{\mathsf{NR}_{\mathsf{g}}(y) + 1 \mid y \trianglelefteq^{g} x\}$$

With $\mathsf{NR}_{\mathsf{g}}(0) = 0$

Remark 29. *This generalized notion of nested truncation can be trivial, or otherwise, close to trivial. See the following example.*

Example 4. *Let* $\mathsf{g} = \chi_{\geq 0}$*. The nested sgn-truncation rank is precisely the one induced by* $\trianglelefteq$*, since* $\trianglelefteq_n^{sgn} = \trianglelefteq$ *for all* n*. This follows because for all* $a, b \in \mathbb{J}^*$*,* $g(a), g(b) = \pm 1$*, so for* $x \trianglelefteq_1^{g} y$*, we* $a \trianglelefteq b$ *and* $z, w \in \text{No}$ *and* $r \in \mathbb{R}^{\times}$ *such that*

$$x = z + sgn(r)g(a) = z \pm 1$$

$$y = z + rg(b) + w = z + \pm r + w$$

are in standard form.

But this means w *must be 0 or an infinitesimal, and* $x \trianglelefteq y$*. Proceeding by induction, we find* $\trianglelefteq_n^{g} = \trianglelefteq$ *for all* n*.*

CHAPTER 6

EXAMPLES OF GENETIC FUNCTIONS AND VEBLEN RANK

So far we have treated genetic functions in the abstract setting, with only a few concrete examples. In this section we will cover several fundamental entire genetic functions of interest, among them exp and arbitrary real-analytic functions via their power-series expansion. We then show how to extend the notion of Veblen rank to functions that are effectively defined only on the convex Class of positive numbers, e.g. log, which we can then use to study the κ and λ functions which can be given uniform recursive definitions, but are nonetheless not entire genetic in the sense constructed in Chapter 3. Finally, we conclude by examining how to describe the Veblen rank ∂_{BM}.

6.1 Ordinal functions

Two very important maps that will appear throughout this dissertation are the ω maps and the ϵ maps:

Definition 67. *For any* $a \in \mathrm{No}$

$$\omega(a) := \left\{0, \omega(a^L) \cdot n\right\} \mid \left\{\frac{\omega(a^R)}{2^n}\right\}$$

We set $\omega_0(a) = a$ *and* $\omega_{n+1}(a) = \omega(\omega_n(a))$. *Then we can define*

$$\epsilon(a) := \left\{\omega^{(n)}(a), \omega^{(n)}(\epsilon(a^L + 1))\right\} \mid \left\{\omega^{(n)}(\epsilon(a^R) - 1)\right\}$$

Following Theorem 51, every Veblen function φ_α has Veblen rank α.

6.2 Primitive Characteristic functions and other definable functions

Recall earlier that we had defined $\chi_{\geq 0}$ as a genetic function with a rational function for the Left option. We can define arbitrary step-functions as follows:

Definition 68. *For all* $a, b \in \text{No}$,

$$\chi_{\geq a} = \left\{ \frac{x - a}{1 + (x - a)^2} \right\} \mid \{\emptyset\}$$

$$\chi_{>a} = \left\{ \frac{x^L - a}{1 + (x - a)^2} \right\} \mid \{\emptyset\}$$

$$\chi_{<a} = 1 - \chi_{\geq a}$$

$$\chi_{\leq a} = 1 - \chi_{>a}$$

$$\chi_{=a} = \chi_{\leq a}\chi_{\geq a}$$

$$\chi_{\neq a} = \chi_{<a} + \chi_{>a}$$

and if X *is a finite set of points* $x_0, \dots x_n$

$$\chi_X = \sum_{i \in [n]} \chi_{x_i}$$

Further, for $I = [a, b]$*, we can define* $\chi_I = \chi_{\geq a} \chi_{\leq b}$*, and similarly for the half-open and open interval cases.*

Since each χ map above has the uniformity property and vacuously satisfies the order properties, we have that all simple step functions above are genetic. Furthermore, given $(I_j)_{j \in [n]}$, and g_j are also genetic functions, then

$$f = \sum_{j \in [n]} g_j \chi_{I_j}$$

is genetic. Notably these definable functions corresponding to some definable set E allow us to meaningfully describe the complexity of definable set in terms of the corresponding characteristic function's Veblen rank. Further, because the step-wise characteristic function is Veblen rank 0, the Veblen rank of χ_E will be immediately determined by the most complicated number appearing in a *term* defining E.

Corollary 12. *Suppose* E *is a Class definable by some quantifier free formula* $\phi(x)$ *where* ϕ *is a formula expressed in* $\mathcal{L}_{\text{oring}} \cup \mathcal{G}$ *for some set of genetic functions* $\mathcal{G}$ *(including constant functions). Then* $\mathsf{VR}(\chi_E) = \mathsf{VR}(\phi) = \mathsf{VR}(E)$*.*

We are also able to define max functions on two variables as follows:

Definition 69. *Let*

$$M(x,y) = \left\{x^L, y^L\right\} \mid \left\{x^R, y^R\right\}.$$

It is immediate that

Proposition 20. $M(x,y)$ *has Veblen rank 0.*

We can further define $\dot{-}$ via

Definition 70.

$$x \dot{-} y = (x + (-y)) \cdot \chi_{\geq 0}(x + (-y))$$

which also immediately satisfies

Proposition 21. $VR(\dot{-}) = 0$

Finally, using this we can define a minimum as a genetic function as follows:

Definition 71. $m(x,y) = x(\chi_{>0}(y \dot{-} x)) + y(\chi_{>0}(x \dot{-} y)) + x(\chi_{=0}(y \dot{-} x))$

and again, it is immediate that

Proposition 22. *VR(m(x,y))=0*

6.3 exp, log, and the g function

Definition 72. *For each* $x \in \text{No}$ *and* $n \in \omega$*, let*

$$[x]_n := \sum_{i \leq n} \frac{x^n}{n!}$$

We define $\exp : \mathrm{No} \to \mathrm{No}_{>0}$ *by*

$$\exp(x) := \left\{0, \exp(x^L)[x - x^L]_n, \exp(x^R)[x - x^R]_{2n+1}\right\} \mid \left\{\frac{\exp(x^R)}{[x^R - x]_n}, \frac{\exp x^L}{[x^L - x]_{2n+1}}\right\}$$

where we take care to omit the cases from the Left and Right options where $[x - x^R]_{2n+1}$ *and* $[x^L - x]_{2n+1} < 0$*. For example, in the case of the Left options, this can be achieved by substituting* $\exp(x^R)[x - x^R]_{2n+1}$ *with* $\exp(x^R)\chi_{>0}([x - x^R]_{2n+1})[x - x^R]_{2n+1}$*, and in the case of the Right options, substituting* $\frac{\exp x^L}{[x^L - x]_{2n+1}}$ *with* $\frac{\exp x^L}{(\chi_{>0}([x^L - x]_{2n+1})[x^L - x]_{2n+1}) + \chi_{<0}([x^L - x]_{2n+1})]}$.

The following facts summarize the main results of Chapter 10 of [1]:

Fact 5.

1. $\exp$ *is a monotonic function onto* $\mathrm{No}_{>0}$*;*
2. $\exp \restriction \mathbb{R}$ *is the real exponential function.*
3. $\exp(x+y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathrm{No}$*; furthermore,* $\exp$ *is an isomorphism between ordered abelian groups* $(\mathrm{No}, +, <)$ *and* $(\mathrm{No}_{>0}, \cdot, <)$.
4. $\exp$ *is not a* $<_s$*-hierarchy preserving map.*
5. *For* $x \in \mathrm{No}_{>0}$*,* $\exp(\omega(x)) = \omega(\omega(g(x)))$*, where* $g : \mathrm{No}_{>0} \to \mathrm{No}$ *defined by*

$$g(x) := \left\{c(x), g(x^L)\right\} \mid \left\{g(x^R)\right\},$$

with $c(x)$ *the unique ordinal such that* $\omega(c(x)) \sim x$*. Further*

- *for all* $n \in \omega$*,* $g(n) = n$*;*

- *for all* $\alpha \in \textsc{On}$, *if there exists* $\gamma \in \textsc{On}$ *so that* $\epsilon(\gamma) \leq \alpha < \epsilon(\gamma)+\omega$, *then* $g(\alpha) = \alpha+1$, *otherwise* $g(\alpha) = \alpha$;
- *if* $\beta \in \textsc{On}$ *and* n *is a positive integer, then* $g(2^{-n}\omega^{-\beta}) = -\beta + 2^{-n}$
- *if* $x \geq 1, g(x) \geq x$;
- *if* $x \in [\epsilon(\alpha) + \omega, \alpha]$ *such that* $\alpha \in \epsilon(\alpha + 1)$, *then* $g(x) = x$;
- *if* $x \in [1, \alpha]$, *such that* $\alpha \in \epsilon(0)$, *then* $g(x) = x$.
- *for all x, if* y *such that* $y = \omega(y)$ *and* $y \leq x \leq y + n$ *for some* $n \in \omega$, *then the sign sequence of* (x) *is the sign sequence* (y) *concatenated by the sequence* S, *and the sign sequence of* $g(x)$ *is* $(y) \frown \langle 1, 0\rangle \frown S$.
- $g(x) = x$ *if and only if* x *is one of the following two forms:*

  (a) x *is less than some ordinal* $\alpha < \epsilon_0$, *and* $x > \frac{n}{\omega}$ *for all integers* n.

  (b) the sign sequence of x *begins with at least* ϵ_0 *many pluses and the first string in the sequence such that initial segment of* x *which terminates at the end of the string is not a generalized epsilon is a string of pluses. Furthermore, if* α *is the number of pluses, choose*[1] ϵ *to be the largest number of pluses such that* $\epsilon \leq \alpha$ *and the sequence obtained by replacing the final string of* α *pluses by* ϵ *pluses is a generalized epsilon number. Then* $x > \epsilon + n$ *for all integers by* n.

Every fact pertaining to function $g(x)$ follows by induction from the genetic definition Gonshor presents, which is the content of Theorem 10.13 of [1], provided below in full:

[1] *Such an* ϵ *exists since the least upper bound of an epsilon number is also an epsilon number.*

Theorem 56. *Letting* $a = \sum_{\alpha} \omega^{a_i} r_i$*, we set* $c = a_0$*, i.e.* c *is the unique surreal number such that* $a \sim \omega^c$*. Then*

$$g(a) := \{c, g(a_L)\} | \{g(a_R)\}$$

Proof. From our earlier work, we have for positive surreal numbers x that

$$\exp(\omega^x) = \omega^{g(x)}$$

so we identify $G(a) = \omega^{g(a)}$ by the theorem where g was introduced.

Now, we define $G(0) = 0$, and then from the proof where g was introduced, by cofinality we find that

$$G(a) := \{rG(a_L) + na\} | \{sG(a_R)\} = \{0, rf(a_L) + na\} | \{sf(a_R)\}.$$

Then, by inductively substituting $G(a^o) = \omega^{g(a^o)}$, we obtain

$$G(a) = \left\{0, na, r\omega^{g(a_L)} + na\right\} | \left\{s\omega^{g(a_R)}\right\}.$$

Since na will be equicofinal with $n\omega^c$ and $r\omega^{g(a_L)}+na$ will be equicofinal with $r\omega^{g(a_L)}+n\omega^c$ and thus equicofinal with

$$n\omega^{\max(g(a_L),c)},$$

we find that

$$G(a) = \left\{0, n\omega^{c}, n\omega^{\max(g(a_L),c)}\right\} \mid \left\{s\omega^{g(a_R)}\right\}.$$

Thus, with $G(a) = \omega^{g(a)}$, by our definition of the ω map and cofinality we have

$$g(a) = \{c, \max(g(a_L), c)\} \mid \{g(a_R)\} = \{c, g(a_L)\} \mid \{g(a_R)\}$$

□

Example 5. *We quickly verify that* $\exp(\omega^{\frac{1}{\omega}}) = \omega$ *using* g *as follows. By induction, assume that* $g(2^{-n}) = 2^{-n}$. *Then*

$$g(2^{-n-1}) = g(\{0\} \mid \left\{2^{-n}\right\}) = \{0\} \mid \left\{g(2^{-n})\right\} = \{0\} \mid \left\{2^{-n}\right\} = 2^{-n-1}$$

whence

$$g(\omega^{-1}) = g(\{0\} \mid \left\{2^{-n}\right\}) = \{-1\} \mid \left\{2^{-n}\right\} = 0.$$

Thus

$$\exp(\omega^{\frac{1}{\omega}}) = \omega^{\omega^{0}} = \omega.$$

Sharp-eyed readers may immediately ask whether the definition

$$g(a) := \{c, g(a_L)\} \mid \{g(a_R)\}$$

is genetic in the sense of Chapter 3. Immediately speaking, it is not, since it is not an entire genetic function. The main obstacle is showing that $c(x)$ is not an entire genetic function (since there is no surreal number such that 0 and ω^c inhabit the same Archimedean equivalence class).

We will return to g when we study log and a corresponding entire genetic function $h(x)$ such that $g(x) = h^{-1}(x)$, using Lemma 14. We now state and summarize proof of the following generalized linearity property for exp from [1].

Theorem 57. *If* $a_i > 0$ *for all* $i \in \alpha$*, then*

$$\exp(\sum_{\alpha} \omega^{a_i} r_i) = \omega^y$$

where

$$y = \sum_{\alpha} \omega^{g(a)_i} r_i$$

with $g(a)_i = g(a_i)$*.*

Proof. Since $\exp x$ and ω^x are both homomorphisms, this follows immediately for all finite sums and rational r_i. From here, we proceed in stages.

First, for monomials $\omega^a r = \{\omega^a r_L\} | \{\omega^a r_R\}$, where r^o are given as some dyadic representation, by induction and the density of the dyadic representations in $\mathbb{R}$, we have that

$$\exp(\omega^a r) = \{0, \exp(\omega^a r_L)_n[\omega^a r - \omega^a r_L]\} | \left\{ \frac{\exp(\omega^a r_R)}{{}_n[\omega^a r_R - \omega^a r]} \right\}.$$

We then simplify the representatives by mutual cofinality to

$$\exp(\omega^a r) = \left\{0, \omega^{\omega^{g(a)} r_L + na}\right\} \mid \left\{\omega^{\omega^{g(a)} r_R - na}\right\}.$$

Hence, we have

$$\omega^{\omega^{g(a)}} > \omega^{na}$$

from which

$$\omega^{g(a)} > na$$

follows in general for all positive integers. Thus

$$\omega^{g(a)} > \frac{n}{r - r_L} a \equiv \omega^{g(a)} r - \omega^{g(a)} r_L > na,$$

whence

$$\omega^{g(a)} r_R - na > \omega^{g(a)} r > \omega^{g(a)} r_L + na.$$

Having satisfied the inbetweenness condition and since the lower terms have no maximum and the upper terms have no minimum, by cofinality we find that

$$\omega^{\omega^{g(a)} r} := \left\{0, \omega^{\omega^{g(a)} r_L}\right\} \mid \left\{\omega^{\omega^{g(a)} r_R}\right\}.$$

We now proceed to induct on α for arbitrary sums.

The non-limit cases follow immediately by the additive properties of the exp and ω maps.

Supposing that α is a limit ordinal, then for arbitrary $\gamma \in \alpha$ and finite $s > 0$,

$$\sum_{\alpha} \omega^{a_i} r_i = \left\{ \sum_{\gamma} \omega^{a_i} r_i - \omega^{a_\gamma} s \right\} | \left\{ \sum_{\gamma} \omega^{a_i} r_i + \omega^{a_\gamma} s \right\},$$

whence

$$\exp(\sum_{\alpha} \omega^{a_i} r_i) = \left\{ 0, \exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{\alpha_\gamma} s)(\omega^{a_\gamma} \sigma)^n \right\} | \left\{ \exp(\sum_{\gamma} \omega^{a_i} r_i + \omega^{a_\gamma})(\omega^{a_\gamma} \rho)^{-n} \right\},$$

where σ (and similarly ρ) is such that

$$\omega^{a_\gamma} \sigma = \omega^{\alpha_\gamma} s + \sum_{\alpha \backslash \gamma} \omega^{a_i} r_i$$

i.e. $|s - \sigma|, |s - \rho|$ will be infinitesimal.

Furthermore,

$$\sum_{\alpha} \omega^{g(a)_i} r_i = \left\{ \sum_{\gamma} \omega^{r_{g(a)} - g(a)} \omega^{g(a_\gamma)} s \right\} | \left\{ \sum_{\gamma} \omega^{g(a)_i} r_i + \omega^{g(a_\gamma)} s \right\},$$

and since the lower terms have no maximum and the upper terms have no minimum, we find that

$$\omega^{\sum_{\alpha} \omega^{g(a)_i} r_i} = \left\{ 0, \omega^F \right\} | \left\{ \omega^G \right\}$$

where F, G are the set of lower and upper terms respectively.

As is common in all of these proofs, we will use cofinality to show that the representation of $\exp(\sum_{\alpha} \omega^{a_i} r_i)$ will give $\omega^{\sum_{\alpha} \omega^{g(a)_i} r_i}$ after first verifying the betweenness condition.

The betweenness condition follows by mutual cofinality and several obvious substitutions such as $\omega^{g(a)} > na$ for all $n \in \mathbb{Z}$, and from s not being an infinitesimal. Specifically, a common lower term will be

$$\exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{a_\gamma} s)\omega^{na_\gamma} = \omega^{y}$$

where $y = \sum_{\gamma} \omega^{g(a)_i} r_i - \omega^{g(a_\gamma)} s + na_\gamma$ by the inductive hypothesis and the additivity of exp. We then see the betweenness for lower terms is satisfied as

$$\omega^{y} < \sum_{\gamma} \omega^{g(a)_i} r_i - \omega^{g(a_\gamma)} \frac{s}{2} < \sum_{\gamma} \omega^{g(a)_i} r_i,$$

and a similar inequality holds for the upper terms, so that by the inductive hypothesis, a typical term of ω^{F} is of the form $\exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{\alpha_\gamma} s)$. Since $a > 0$ by hypothesis, we have that $\omega^{na} \geq 1$ and this completes the proof for representatives of ω^{F}. A similar argument is run for ω^{G}. □

Remark 30. *As a consequence of this result, studying the complexity of* $\exp x$ *amounts to studying the growth of* g*. In particular, these results are essential for proving that* $\mathsf{VR}(\exp) = 0$.

We conclude this section by providing an explicit genetic definition for *log*:

Definition 73. *We will first define* $\log \circ \omega$ *as an entire-genetic function:*

$$\log \circ \omega(x) = \log(\omega^x) = \left\{ \log(\omega^{x^L} + n), \log(\omega^{x^R}) - \omega^{\frac{x^R - x}{n}} \right\} \mid \left\{ \log(\omega^{x^R} - n), \log(\omega^{x^L} + \omega^{\frac{x - x^L}{n}}) \right\},$$

where n *runs through all positive integers. By Theorem 10.8 of [1], one finds that* $\log(x)$ *is defined for all positive surreal numbers* x*, with Corollary 10.3 of [1] confirming that* $\exp$ *is onto the class of all positive surreal numbers and* $\log$ *as the appropriate inverse function.*

In practice, for any $x \in \mathrm{No}_{>0}$*, we first factor the Conway normal form into*

$$x = \omega^{\ell(x)} r_0 (1 + \epsilon),$$

and then compute

$$\log(x) = \log(\omega^{\ell(x)}) + \log(r_0) + \log(1 + \epsilon),$$

where $\log(r_0)$ *is the standard log for the real numbers and* $\log(1+\epsilon)$ *is given by the power series expansion*

$$\log(1 + \epsilon) = \sum_{n \geq 1} \frac{(-1)^{n+1} \epsilon^n}{n},$$

given that ϵ *is an infinitesimal number.*

Proposition 23. $\mathsf{VR}(\log \circ \omega) = 0$*.*

Proof. The image of ω forms the Class of leaders in the surreal number with respect to the Conway normal form, we map into the Convex class of all non-negative surreal numbers, and so we can evaluate log as the inverse of exp on these values. Furthermore, in both cases the Veblen rank will remain 0 as ω sends all numbers of length below any epsilon number ϵ to a number of length below ϵ, and log has Veblen rank 0 immediately by Lemma 14. □

Remark 31. *While* log *itself is not an entire genetic function, one can still reason about* $\log \circ \omega$ *as an entire genetic function with the notion of Veblen rank. Furthermore, as we see later on in this section,* exp *and* log *iterates can be used to define entire genetic functions* κ *and* λ*, which in turn can be used to define the* Berarducci-Mantova *derivative, which endows the surreal numbers with a real differential algebraic structure.*

While log is interesting in its own right, Gonshor introduces the definition of log to study growth properties of exp, and in particular, to establish that exp is the correct notion of an natural exponential operation for the surreal numbers. In particular, the growth rate of log can be studied with a genetic function $h : \mathrm{No} \to \mathrm{No}_{>0}$ which is the natural inverse to the g function used for studying the growth rate of exp, as seen with the following definition summarizing several results in Ch 10 of [1].

Definition 74. *Let* $a \in \mathrm{No}^{\times}$ *have the following Cantor normal form:* $\sum\limits_{\nu a} \omega^{a_i} r_i$.

Then

$$\log(\omega^{a}) = \log\left(\prod_{\nu a} (\omega^{\omega^{a_i}})^{r_i}\right) = \sum_{\nu a} \omega^{h(a_i)}$$

where

$$h(a) = \left\{0, h(a^L)\right\} \mid \left\{h(a^R), \frac{\omega^{a}}{2^n}\right\}.$$

Many of the results listed above as Facts about exp *and* g *are consequences of* $h = g^{-1}$.

Proposition 24. *The function*

$$h(a) = \left\{0, h(a^L)\right\} \mid \left\{h(a^R), \frac{\omega^a}{2^n}\right\}$$

is genetic in the sense of Chapter 3. In particular, it is an entire recursively defined function satisfying the order and cofinality properties.

Proof. The order properties follow immediately by induction, and the proof of cofinality follows from the proof of uniformity found in Ch 10 [1]. □

Proposition 25. $VR(h) = 0$ *and consequently* $VR(g) = 0$.

Proof. By Lemma 3.1 of [35], we have

$$\iota(h(x)) \leq \omega^{\iota x+1}$$

for all $x \in \mathrm{No}$, whence we have $VR(h, \gamma) = 0$ for all γ, from which we conclude that $VR(h) = 0$. Then following Lemma 14, and the content of the proof of Theorem 10.11 from [1] which establishes $g = h^{-1}$,[1] we have that $VR(g) = 0$. □

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[1]Specifically, for all $x \in \mathrm{No}$, given that g is related to exp and ω via $\exp(\omega(x)) = \omega(\omega(g(x)))$, and h is related to exp and ω by $\exp(\omega(h(x))) = \omega(\omega(x))$, we have $g = h^{-1}$.

Having studied exp and g implicitly with the application of log and h, it is natural to ask if we can study log as a genetic function. We give above an inductive construction for log, but before we can conclude log is genetic on the convex class of positive numbers, we need to verify the uniformity property holds.

Theorem 58. *The uniformity theorem is valid for the natural log function.*

Proof. We summarize the proof, which can be found in [1]. The following inequalities are derived from standard order of magnitude arguments and properties we have established about the ω map: For lower elements $a_L < x < a$, we have $\log(\omega^x) + n \geq \log(\omega^{a_L}) + n$ and $\log(\omega^x)+\omega^{\frac{a-x}{n}} \leq \log(\omega^{a_L})+\omega^{\frac{a-a_L}{n}}$. For upper elements $a < x < a_R$,$\log(\omega^x)+n \leq \log(\omega^{a_R})+n$ and $\log(\omega^x) - \omega^{\frac{x-b}{n}} \geq \log(\omega^{b_R}) - \omega^{\frac{a_R-a}{n}}$. As with almost all uniformity theorem proofs, the rest of the proof is handled by the use of the inverse cofinality theorem and an application of the cofinality theorems after these inequalities have been established. □

The details of the proof of Theorem 58 can be filled out with the following facts, summarizing several Theorems and Corollaries of Chapter 10.B and 10.C in [1].

Fact 6.

1. *For all* $a > b$, $\log(\omega^a) - \log(\omega^b) \in \mathrm{No}^{>0}_{>0}$;
2. *For all* $a > b$, $\log(\omega^a) - \log(\omega^b) < \omega^{\frac{a-b}{n}}$ *for all positive integers* n*;*
3. *For* $a > 0$, $\log(\omega^a) < \omega^{\frac{a}{n}}$*;*
4. *For all* a, $\exp(\log(\omega^a)) = \omega^a$*;*

5. *For all* a*,* $\log(\omega^{\omega^{a}})$ *is a power of* ω *(as seen in the previous definition where we introduced the* h *functions).*

6.4 The log-atomic numbers: κ and λ maps

Following [7, 11, 20], we also describe two additional maps necessary for studying log-atomic numbers and the Berarducci-Mantova derivative. While the Berarducci-Mantova derivative will not be genetic in the sense described in this paper, several of the related functions are, and have properties that are worth investigating further.

First, the log-atomic numbers are defined as follows:

Definition 75. *Let* $a \in \mathrm{No}^{>0}_{>0}$*, i.e.* a *is a positive infinite surreal number. We say* a *is* ***log-atomic*** *if for all* $n \in \mathbb{N}$*, there is a* $b_n \in \mathrm{No}$ *such that for the n-fold iterate of* log *we have*

$$\log^{(n)}(a) = \omega^{b_n}.$$

We denote the class of log-atomic numbers by $\mathbb{L}$*.*

Proposition 26. *Let* $\mathbb{E} = \epsilon''\mathrm{On}$ *denote the class of epsilon numbers. Then* $\mathbb{E} \subsetneq \mathbb{L}$

Proof. This follows from the Fact 6. □

The κ numbers are intended to convey a notion of magnitude with respect to the growth of the exp iterates. The authors of [7] define the following relation:

Definition 76. *For any two* $x, y \in \mathrm{No}$ *such that* $x, y > \mathbb{N}$*:*

1. $x \preceq^{\kappa} y$ *if* $x \leq \exp^{(n)}(y)$ *for some* $n \in \mathbb{N}$*;*

2. $x \prec^{\kappa} y$ *if* $x < \log^{(n)}(y)$ *for all* $n \in \mathbb{N}$*;*

3. $x \asymp^{\kappa} y$ *if* $\log^{(n)}(y) \leq x < \exp^{(n)} y$ *for some* $n \in \mathbb{N}$*.*

We say that x *and* y *belong to the same* ***exp-log class*** *if* $x \asymp^{\kappa} y$*.*

Proposition 27. $\asymp^{\kappa}$ *is an equivalence relation.*

Proposition 28. *For all* $x, y \in \mathrm{No}$*, with* $x, y > \mathbb{N}$*,* $x \asymp^{\mathbb{L}} y$ *implies* $x \asymp^{\kappa} y$*.*

We then properly define the κ numbers with respect to a genetic function that identifies canonical representatives of each $\asymp^{\kappa}$ equivalence Class:

Definition 77. *For all* $x \in \mathrm{No}$*,*

$$\kappa(x) := \left\{\exp^{(n)}(0), \exp^{(n)}(\kappa(x^L))\right\} \mid \left\{\log^{(n)}(\kappa(x^R))\right\}$$

where n *ranges over* $\mathbb{N}$*.*

κ numbers are the *simplest* element in their respective $\exp - \log$ class.

Remark 32. *It is seen immediately that* $\kappa(0) = \omega(0)$ *and* $\kappa(1) = \varepsilon(0)$*.*

Proposition 29. $\mathsf{VR}(\kappa) = 1$

Proof. Using our knowledge that the image of κ will lie in $\mathrm{No}_{>0}$ and that when restricted to this space, we can intelligibly say that $\mathsf{VR}(\log) = \mathsf{VR}(\exp) = 0$, it suffices to check by induction on complexity, that for each γ, we have $\mathsf{VR}(\kappa, \gamma) \leq 1$.

In particular, this amounts to checking that for all γ, for all $x \in \mathrm{No}(\epsilon_{\gamma})$, $\sqrt{\kappa(x)} < \varphi_2(\gamma)$.

Towards that end, we recall Lemma 5.2 of [6], for all $x \in \mathbf{No}$

$$\iota(\exp(x)) \leq \omega^{\omega^{2\iota(x)+3}}$$

and Lemma 5.4 of [6] for positive x

$$\iota(\log x) \leq \omega^{\omega^{3\iota(x)+3}}.$$

By induction on ιx, when $x = 0$, we have $\kappa(0) = \omega$ and further that $\kappa(1) = \epsilon_0$, so we immediately have that $\mathsf{VR}(\kappa, 0) \geq 1$. Similarly, $\kappa(-1)$ is also at most length $\epsilon(0)$ by ω many applications of Lemma 5.4.

We now want to show that $\mathsf{VR}(\kappa, 0) \not\geq 2$. But by transfinite applications of the two Lemmas, we find for all $\iota x < \varphi_1(0)$ that

$$\sqrt{\kappa(x)} \leq \varphi_1(\iota x) < \varphi_1(\varphi_1(0)) < \varphi_2(0),$$

whence we find $\mathsf{VR}(\kappa, 0) = 1$.

Further, by induction, for all $\gamma \neq \varphi_3(\delta)$ for any ordinal δ, we will have $\varphi_1(\varphi_1(\gamma)) < \varphi_2(\gamma)$ hold, whence $\mathsf{VR}(\kappa, \gamma) = 1$.

Finally for cases where $\gamma = \varphi_3(\delta)$ for some ordinal δ, as a fixed point of both φ_1 and φ_2, we note that inequality still holds since in any possible limit, we will still have $\varphi_1(\iota x) < \gamma$.

$\square$

Fact 7. *1.* $x \leq_s y$ *if and only if* $\kappa(x) \leq_s \kappa(y)$.

2. *For all* $x > \mathbb{N}$, *there exists* $\kappa(y) \leq_s x$ *such that* $\kappa(y) \asymp^{\kappa} x$, *so each* $\kappa(y)$ *is the simplest element in its respective equivalence Class.*

3. $x < y$ *implies that* $\kappa(x) \prec^{\kappa} \kappa(y)$.

4. $\log^{(n)}(\kappa(x))$ *is always of the form* $\omega(\omega(y))$, *and therefore each* $\log^{(n)}(\kappa(x)) \in \mathfrak{M}$.

5. $\kappa(\mathrm{No}) \subset \mathbb{L}$.

6. *There are numbers in* $\mathbb{L}$ *which cannot be obtained from* $\kappa(\mathrm{No})$ *by finitely many applications of* log *and* exp

Following this last fact, with the goal of generating $\mathbb{L}$ from $\kappa(\mathrm{No})$, Berarducci and Mantova focus on the $\kappa(-\alpha)$ numbers for $\alpha \in \mathrm{On}$. Specifically

$$\kappa(-\alpha) = \mathbb{N}|\{\log^{(n)}(\kappa(-\beta) \mid n \in \mathbb{N}, \beta \in \alpha\}$$

will be the simplest positive number less than $\log^{(n)}(\kappa(-\beta))$ for all $n \in \mathbb{N}$ and $\beta \in \alpha$. From this, they find

Proposition 30. *The sequence* $\langle \kappa(-\alpha) \mid \alpha \in \mathrm{On} \rangle$ *is a decreasing and coinitial with the positive infinite numbers (i.e. every positive infinite number is greater than some* $\kappa(-\alpha)$, *and from this we find* $\mathbb{L}$ *is coinitial in the positive infinite numbers.*

In general, a finer notion borrowed from the study of Hardy fields is used to study $\mathbb{L}$.

Definition 78. *Given* $\mathfrak{a}, \mathfrak{b} \in \mathrm{No}^{>0}_{>0}$*, we say* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have the same* ***level*** *if there exists* $\mathfrak{n}$ *such that* $\log^{(\mathfrak{n})}(\mathfrak{a}) \sim \log^{(\mathfrak{n})}(\mathfrak{b})$*, i.e. if for some natural number* $\mathfrak{n}$*,* $\log^{(\mathfrak{n})}(\mathfrak{a})$ *and* $\log^{(\mathfrak{n})}(\mathfrak{b})$ *belong to the same Archimedean class. We say* $\mathfrak{a}$ *is a* λ*-number if it is the simplest number in its own level. We parametrize this with the partial genetic map:*

$$\lambda(x) = \left\{\mathfrak{m}, \exp^{(\mathfrak{n})}(\mathfrak{n} \cdot \log^{(\mathfrak{n})} \lambda(x^L)\right\} \mid \left\{\exp^{(\mathfrak{n})}\left(\frac{1}{\mathfrak{m}} \log^{(\mathfrak{n})} \lambda(x^R)\right)\right\}$$

as $\mathfrak{m}, \mathfrak{n} \in \omega$*.*

Remark 33. *A quick calculation shows the following:*

- $\lambda(0) = \omega$*;*
- $\lambda(1) = \exp(\omega)$*;*
- *and* $\lambda(\omega) = \epsilon_0$*.*

Berarducci-Mantova [7] proved the following:

Theorem 59. *All* λ*-numbers are log-atomic, and all log-atomic numbers are* λ *numbers.*

Following the proof of Proposition 29

Proposition 31. $\mathsf{VR}(\lambda) = 1$

Proof. Following the remark above, we see that $\mathsf{VR}(\lambda, 0) \geq 1$ since $\lambda(2) = \epsilon_0$.

Using our bounds on the growth of n-iterates of log and exp, it will suffice to establish the following inequality for all x such that $\iota x < \epsilon_\gamma$:

$$\iota\lambda(x) < \varphi_1(\iota x) < \varphi_2(\gamma).$$

Supposing

$$\iota\lambda(x) < \varphi_1(\iota x)$$

for all $\iota x < \alpha < \varepsilon_0$, when $\iota x = \alpha$, where it suffices to examine the following two pairs of options:

<u>Left</u> $\exp^{(n)}(m \log^{(n)} \lambda(x^L))$

<u>Right</u> $\exp^{(n)}(\frac{1}{m} \log^{(n)}(\lambda(x)^R))$

It will further suffice to only consider the Right case, as the argument for the Left case is identical.

Using Corollary 4.3 of [6], what we've referred to elsewhere as the *weak product lemma*, $\iota(ab) \leq \omega[\iota(a)]^2[\iota(b)]^2$, by our induction hypothesis we have both

$$\iota(\frac{1}{m} \log^{(n)}(\lambda(x)^R) \leq \omega \cdot \omega^2 \cdot [\iota(\log^{(n)}(\lambda(x^R))]^2$$

and

$$\iota\lambda(x^R) < \varphi_1(\iota(x^R)) < \varphi_1(\alpha),$$

since φ_i are all strictly monotonically increasing, and so

$$\iota(\log^{(n)}(\lambda(x^R))) \leq \omega^{\cdot^{\cdot^{\cdot^{\omega^{3\iota(x^R)+3}}}}}_{(2n-2)} < \omega^{\cdot^{\cdot^{\cdot^{\omega^{\varphi_1(\iota x^R)}}}}}_{(2n-2)} = \varphi_1(\iota(x)) < \varphi_1(\alpha)$$

for all $n \in \omega$, since

$$3\iota(x^R) + 3 < \varphi_1(\iota x^R)$$

because all non-rank 0 Veblen hierarchy ordinals are additively and multiplicatively indecomposable. Further,

$$\iota(\frac{1}{m}\log^{(n)}(\lambda(x)^R) \leq \omega \cdot \omega^2 \cdot [\iota(\log^{(n)}(\lambda(x^R))]^2 < \varphi_1(\alpha)$$

for all $m, n \in \omega$.

We run this tower of powers argument again for $\exp^{(n)}$, with $a_{n,m}(x^R) = \frac{1}{m}\log^{(n)}(\lambda(x^R))$

$$\iota(\exp^{(n)}(a_{n,m}(x^R)) \leq \omega^{\cdot^{\cdot^{\cdot^{\omega^{2\iota(a_{n,m}(x^R))+3}}}}}_{(2n-2)} < \omega^{\cdot^{\cdot^{\cdot^{\omega^{\varphi_1(\alpha)}}}}}_{(2n-2)} = \varphi_1(\alpha) < \varphi_2(0)$$

since

$$2\iota(a_{n,m}(x^R)) + 3 < \varphi_1(\alpha).$$

This establishes that $\mathsf{VR}(\lambda, 0) = 1$ since $\mathsf{VR}(\lambda, 0) \not\geq 2$.

Following the argument from Proposition 29, when we consider $\gamma > 0$, we break into the two cases depending on whether γ is a fixed point of φ_2. As in Proposition 29, the inequalities derived still hold, and we find that

$$\mathsf{VR}(\lambda, \gamma) \leq 1,$$

whence $\mathsf{VR}(\lambda) = 1$. □

As a consequence of Propositions 29 and 31, we have the following result:

Theorem 60. *For all $\gamma \in \mathrm{ON}$, $A_\gamma = \mathrm{NO}(\varphi_2(\gamma))$, we have that A_γ is closed with respect to exp-log classes, and log-atomic numbers.*

6.4.0.1 $\underline{\partial_{\mathrm{BM}}}$

While the following lies outside the intended scope of this dissertation, as the derivative defined is not an entire genetic function as we have defined it in Chapter 3.1, given Theorem 60, it is nonetheless included for the benefit of interested readers. In [7] Berarducci and Mantova provide a construction of a derivative ∂_{BM} such that $(\mathrm{NO}, +, \cdot, \exp, \partial_{\mathrm{BM}})$ is a Hardy type series derivation. More precisely, NO can be equipped with a derivation so that NO is a Liouville closed H-field such that ∂_{BM} is surjective and sends infinitesimals to themselves.

We provide the definitions of surreal pre-derivations $\mathrm{D}_{\mathbb{L}}$ and surreal derivations D in such a way to make $(\mathrm{NO}, \mathrm{D})$ an H-field, a generalized notion of a Hardy field. Afterwards, we will provide the definition of the Berarducci-Mantova derivative, and then explore some immediate facts and properties of the derivative.

Definition 79. *A **(surreal) pre-derivation** is a map $\mathrm{D}_{\mathbb{L}} : \mathbb{L} \to \mathbb{R}_{>0}\mathfrak{M}$ such that*

1. $\log(D_{\mathbb{L}}(\lambda)) - \log(D_{\mathbb{L}}(\mu)) \prec \max\{\lambda, \mu\}$.

2. $D_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda) D_{\mathbb{L}}(\lambda)$ *for all* $\lambda, \mu \in \mathbb{L}$.

A ***surreal derivation*** *is a function* $D : \mathrm{No} \to \mathrm{No}$ *with the following properties:*

1. *(Leibniz rule):* $D(xy) = D(x) + D(y)$

2. *(Strong Additivity):* $D(\sum_{i \in I} x_i) = \sum_{i \in I} D(x_i)$ *for all summable sequences* $\langle x_i \mid i \in I \rangle$

3. *(Compatibility):* $D(\exp(x)) = \exp(x) D(x)$

4. *(Real constant field):* $\ker(D) = \mathbb{R}$

5. *(H-field): if* $x > \mathbb{N}$, *then* $D(x) > 0$

The following facts are true for all surreal derivations D:

Fact 8. 1. *if* $1 \not\asymp x \succ y$, *then* $D(x) \succ D(y)$*;*

2. *if* $1 \not\asymp x \sim y$, *then* $D(x) \sim D(y)$*;*

3. *if* $1 \not\asymp x \asymp y$, *then* $D(x) \asymp D(y)$

4. *For* $x, y \in \mathrm{No}$, *if* $x, y, x - y$ *are all positive infinite, then*

$$\log(D(x)) - \log(D(y)) \prec x - y \preceq \max\{x, y\}$$

Berarducci-Mantova define their derivation ∂_{BM} first by defining one on $\mathbb{L} \to \mathrm{No}_{>0}$, and then extending the definition to all of No by means of path-derivatives.

Definition 80. *For $\lambda \in \mathbb{L}$, with α ranging over the ordinals, let*

$$\partial_{\mathbb{L}} := \exp\left(-\sum_{\lambda \preceq^{\kappa} \kappa(-\alpha)} \sum_{i=1}^{\infty} \log^{(i)}(\kappa(-\alpha)) + \sum_{i=1}^{\infty} \log^{(i)}(\lambda)\right)$$

Since $\langle \log^{(i)} \lambda \rangle$ is a strictly decreasing sequence of monomials, it is summable. Similarly, $\langle \kappa(-\alpha) \rangle$ is decreasing, so $\langle \log^{(i)}(\kappa(-\alpha))$ will also be summable. Furthermore, if $\lambda = \kappa(-\alpha)$ for some ordinal α, then the terms $\log^{(i)}(\lambda)$ cancel out, and we find that

$$\partial_{\mathbb{L}}(\lambda) = \exp\left(\sum_{\beta<\alpha} \sum_{i=1}^{\infty} \log^{(i)}(\kappa(-\beta))\right)$$

with $\partial_{\mathbb{L}}(\omega(0)) = \partial_{\mathbb{L}}(\kappa(0)) = 1$.

We now define paths and path derivatives, before we define ∂_{BM} with respect to the pre-derivative $\partial_{\mathbb{L}}$.

Definition 81. *A **path** is an sequence $P : \mathbb{N} \to \mathbb{R}^{\times}\mathfrak{M}$ such that for every $n \in \mathbb{N}$, $P(n+1)$ is term of $\ell(P(n))$. $\mathcal{P}(x)$ is the set of paths such that $P(0)$ is a term of x. Given a path P, the **path derivative** $\partial(P) \in \mathbb{R}\mathfrak{M}$ is defined as follows:*

1. *if for some $n \in \mathbb{N}$ such that $P(n) \in \mathbb{L}$, set $\partial(P) = \prod_{i<n} P(i) \cdot \partial_L(P(n))$;*
2. *if for all $n \in \mathbb{N}$, $P(n) \notin \mathbb{L}$, set $\partial(P) = 0$.*

*We define the **Berarducci-Mantova** derivative $\partial : \mathrm{No} \to \mathrm{No}$ by*

$$\partial(x) := \sum_{P \in \mathcal{P}(x)} \partial(P)$$

*Given $x \in \mathrm{No}\backslash\mathbb{R}$, the **dominant path** of x is the path $Q \in \mathcal{P}(x)$ such that $Q(0)$ is the term of maximum non-zero ℓ value of x and $Q(i+1)$ is the leading term of $\ell(Q(i))$ for all $i \in \mathbb{N}$.*

We now state many facts about the pre-derivative, paths, and the Berarducci-Mantova derivative:

Fact 9. 1. *For all $\lambda, \mu \in \mathbb{L}$, $\log(\partial_{\mathbb{L}}(\lambda)) - \log(\partial_{\mathbb{L}}(\mu)) \prec \max\{\lambda, \mu\}$*

2. *For all $\lambda \in \mathbb{L}$, $\partial_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda)\partial_{\mathbb{L}}(\lambda)$*

3. *If P is a path, then $1 \prec P(i+1) \preceq \log(|P(i)|) < P(i)$ for all $i > 0$.*

4. *If $t \preceq u$ are both monomial terms, and v is a term of $\ell(t)$ but not $\ell(u)$, then $v^n \prec \frac{u}{t}$ for all $n \in \mathbb{N}$.*

5. *If P, Q are two paths such that $\partial(P), \partial(Q) \neq 0$, then if $P(0) \preceq Q(0)$ and $P(1)^n \prec \dfrac{Q(0)}{P(0)}$ for all $n \in \mathbb{N}$, then $\partial(P) \prec \partial(Q)$.*

6. *Extending Fact 5, if there exists an n such that for all $m \leq n$, $P(m) \preceq Q(m)$, and $P(n+1)^k \prec \dfrac{Q(n)}{P(n)}$ for all $k \in \mathbb{N}$, then $\partial(P) \prec \partial(Q)$.*

7. *If P,Q are two paths with non-zero path derivative and there exists an $n \in \mathbb{N}$ such that for all $m \leq n$, $P(m) \preceq Q(m)$ and $P(n+1)$ is not a term of $\ell(\mathbb{Q}(n))$, then $\partial(P) \prec \partial(Q)$.*

8. *Given $P \in \mathcal{P}(x)$, $\mathrm{NR}(P(0)) \leq \mathrm{NR}(x)$, and if $\mathrm{NR}(P(0)) = \mathrm{NR}(x)$, then the minimum $\mathfrak{m}$ of $S(x)$ is such that $P(0) = r\mathfrak{m}$ for some $r \in \mathbb{R}^{\times}$.*

9. *Similarly, for all $n \in \mathbb{N}$, $\mathrm{NR}(P(n+1)) \leq \mathrm{NR}(P(n))$ and if equality holds, then there is a minimum $\mathfrak{m}$ in $S(\ell(P(n)))$ such that $P(n+1) = r\mathfrak{m}$ for some $r \in \mathbb{R}^{\times}$.*

10. *For all* $x \in \mathrm{No}$, *there is at most one path* $P \in \mathcal{P}(x)$ *such that* $\mathsf{NR}(P(n)) = \mathsf{NR}(x)$ *for all* $n \in \mathbb{N}$.

11. *If* $x \in \mathrm{No}\backslash\mathbb{R}$ *with dominant path* Q, *then* $\partial(Q) \neq 0$ *and* $\partial(Q)$ *is the leading term of* $\partial(x)$.

12. $\ker \partial = \mathbb{R}$.

13. *If* $x > \mathbb{N}$, *then* $\partial(x) > 0$

14. ∂ *is strongly linear, and therefore strongly additive.*

15. *For all* $\gamma \in \mathbb{J}$, ∂*(*exp*(*γ*))=*exp*(*γ*)*∂*(*γ*)$.*

16. *For all* $x, y \in \mathrm{No}$, $\partial(xy) = x\partial(y) + y\partial(x)$.

17. *For all* $x \in \mathrm{No}$, $\partial(\exp(x)) = \exp(x)\partial(x)$.

Using the facts above, we summarize the proof of summability from [11]

Theorem 61. *For all* $x \in \mathrm{No}$, *the family* $\langle \partial(P) \mid P \in \mathcal{P}(x)\rangle$ *is summable.*

Proof. For any $x \in \mathrm{No}$, it suffices to show that there is no sequence of distinct paths $\langle P_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{P}(x)$ such that we have an infinite ascending chain

$$\partial P_0 \preceq \partial P_1 \preceq \partial P_2 \preceq \cdots ,$$

since $\partial(P) \in \mathbb{R}\mathfrak{M}$ for all $\mathcal{P}(x)$. Towards a contradiction, suppose that there exists such a sequence and let $\alpha = \mathsf{NR}(x)$. Since the paths are distinct, there must be a minimum $m \in \mathbb{N}$ such that $P_i(m) \neq P_j(m)$ for some $i, j \in \mathbb{N}$. We proceed by double induction, first on α, and then on m.

Let $\mathsf{r}\exp(\gamma)$ be the maximum ℓ value from $\{\mathsf{P}_j(0) \mid j \in \mathbb{N}\}$.

By fact 11.8, if $\mathsf{NR}(\gamma) = \alpha$, then $\mathsf{r}\exp(\gamma)$ is also the term of minimum ℓ value, whence $\mathsf{P}_j(0) = \mathsf{P}_0(0)$ for all j. Thus $m > 0$.

If $\mathsf{NR}(\gamma) < \alpha$, we extract a subsequence so that

$$\mathsf{r}\exp(\gamma) = \mathsf{P}_0(0) \succeq \mathsf{P}_1(0) \succeq \mathsf{P}_2(0) \succeq \cdots .$$

If $\mathsf{P}_j(1)$ is not a term of $\gamma = \ell(\mathsf{P}_0(0))$ for some $j \in \mathbb{N}$, but Fact 11.7, we find that $\partial(\mathsf{P}_j) \prec \partial(\mathsf{P}_0)$, which is a contradiction.

Therefore, $\mathsf{P}_j(1)$ must be a term of γ for all $j \in \mathbb{N}$.

Now consider paths Q_j defined by $\mathsf{Q}_j(n) = \mathsf{P}_j(n+1)$, for all $n \in \mathbb{N}$. Let r be the minimum integer such that $\mathsf{Q}_j(r) \neq \mathsf{Q}_k(r)$ for some j, k.

In the case of $\mathsf{NR}(\gamma) = \alpha$, we have that $r = m-1$, and that for all $j \in \mathbb{N}$, we have $\mathsf{Q}_j \in \mathcal{P}(x)$.

Thus, we find that $\partial(\mathsf{P}_j) = \mathsf{P}_j(0) \cdot \partial(\mathsf{Q}_j)$, and that we have a descending sequence

$$\mathsf{P}_0(0) \succeq \mathsf{P}_1(0) \succeq \mathsf{P}_2(0) \succeq \cdots ,$$

from which we derive an ascending sequence

$$\partial Q_0 \preceq \partial Q_1 \preceq \partial Q_2 \preceq \cdots .$$

Now, we either have that (1) NR(γ)=α and $r < m$; or we have (2) $\mathrm{NR}(\gamma) < \alpha$, and both of these contradict the induction hypothesis that no suchh sequence exists in γ.

Thus $\langle \partial P \mid P \in \mathcal{P}(x) \rangle$ is summable. □

Theorem 62. ∂_{BM} *extends* $\partial_{\mathbb{L}}$*.*

Proof. By facts 11.12 to 11.17, we find that ∂_{BM} is a surreal derivation. By restricting ∂_{BM} to $\mathbb{L}$, $\partial_{\mathrm{BM}} \upharpoonright \mathbb{L}$ takes values in the subfield $\mathbb{R}\langle\langle\mathbb{R}\rangle\rangle$ of No. Since we compute ∂_{BM} as finite products of infinite sums, we see that $\partial(\mathbb{R}\langle\langle\mathbb{R}\rangle\rangle) \subset \mathbb{R}\langle\langle\mathbb{R}\rangle\rangle$, from which $\partial_{\mathrm{BM}} \upharpoonright \mathbb{L}\langle\langle\mathbb{L}\rangle\rangle$ will induce an H-field structure on $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$. □

Corollary 13. *Let* $d : \mathbb{L} \to \mathrm{No}_{>0}$ *be a map such that:*

1. *for all* $\lambda, \mu \in \mathbb{L}$*,* $\log(d(\lambda)) - \log(D(\mu)) \prec \max\{\lambda, \mu\}$*;*

2. *for all* $\lambda(\mathbb{L}), d(\exp(\lambda)) = \exp(\lambda) d(\lambda)$*;*

3. $d(\mathbb{L}) \subset \mathbb{R}^{\times}\mathfrak{M}$*.*

Then d *extends to a surreal derivation* D *on* No*.*

CHAPTER 7

$\mathsf{T}_{\mathcal{G}}$

The primary aim of this section is to understand the conditions we need to impose on a set of genetic function symbols $\mathcal{G}$, so that in $\mathcal{L} = \mathcal{L}_{\mathsf{or}} \cup \mathcal{G}$ we can eventually define a homogeneous theory $\mathsf{T}_{\mathcal{G}}$ such that $\mathrm{No} \models \mathsf{T}_{\mathcal{G}}$, and $\mathsf{T}_{\mathcal{G}}$ has both the Joint Embedding Property (JEP) and the Strong Amalgamation Property (SAP). Towards that end we must first do the following:

1. Define $\mathsf{T}_{\mathcal{G}}$ as a first-order theory so that models of $\mathsf{T}_{\mathcal{G}}$ are ordered real closed fields closed under the application of genetic functions $\mathcal{G}$, the cofinality property of the genetic functions is preserved for all option terms (i.e. for all $g \in \mathcal{G}$, and all x, y, z, $y < x < z$ implies that $g^{L}(x, y, z) < g(x) < g^{R}(x, y, z)$), all distinguished structural properties of the genetic functions are preserved (i.e. if they're homomorphisms, additive, etc), and every model contains the ordered subfield $\mathsf{Q}_{\mathcal{G}}$ definable over $\mathcal{L}$.
2. Define a value group $\Gamma_{\mathcal{G}}$ with respect to the fraction field of $(\mathbb{Q} \cup \mathcal{G}^{*}(\mathbb{Q}))$, and use this to define $\mathcal{G}$-structured Hahn fields;
3. Establish that a model $\mathcal{M}$ of $\mathsf{T}_{\mathcal{G}}$ is isomorphic to an initial substructure of No if and only if $\mathcal{M}$ is isomorphic to a truncation-closed, cross-section $\mathcal{G}$-structured subfield of a $\mathcal{G}$-structured Hahn field.

Following item (1), we denote by $\mathsf{T}_{\mathcal{G}}$ as the first order theory satisfied by the following:

Definition 82. *Given a proper set $\mathcal{G} \subsetneq \mathcal{G}$ of entire genetic functions, for an ordered field* K*, let*

$$\mathcal{G}^*[\mathsf{K}] := \mathsf{Frac}(\mathsf{K}[\{g(k) \colon g \in \mathcal{G}^*, k \in \mathsf{K}\})].$$

Specifically, for $\mathsf{K} = \mathbb{Q}$*, we let*

$$\mathcal{G}^{*(0)}(\mathbb{Q}) := \mathsf{Frac}(\mathbb{Q}[\{g(r) \colon g \in \mathcal{G}^*, r \in \mathbb{Q}\})]$$

$$\mathcal{G}^{*(n+1)}(\mathbb{Q}) := \mathcal{G}^*[\mathcal{G}^{*(n)}(\mathbb{Q})]$$

$$\mathrm{Q}_{\mathcal{G}} = \bigcup_{n \in \omega} \mathcal{G}^{*(n)}(\mathbb{Q})$$

We call $\mathrm{Q}_{\mathcal{G}}$ *the* $\mathcal{G}-$***definable field****.*

Proposition 32. $\mathrm{Q}_{\mathcal{G}}$ *is an ordered field.*

Proof. Since $\mathrm{Q}_{\mathcal{G}}$ forms a field, we only need to verify the additional axioms for ordering. To prove $\mathrm{Q}_{\mathcal{G}}$ has an ordering, we establish by induction that each $\mathcal{G}^{*(n)}(\mathbb{Q})$ is an ordered field, which amounts to showing by induction that at each stage the additional ordering axioms are satisfied.

These are satisfied at each stage n via repeated application of the cofinality property defining the Left and Right options for each genetic term in $t \in \mathcal{G}^*$. In particular, because we are able to directly interpret $\mathrm{Q}_{\mathcal{G}}$ as some subfield inside No, $\mathrm{Q}_{\mathcal{G}}$ is ordered at each stage by restricting No to $\mathcal{G}^{*(n)}(\mathbb{Q})$. □

Definition 83. *Given proper set $\mathcal{G} \subset \mathcal{G}$, we define $T_{\mathcal{G}}$ via the following axiom scheme:*

- RCF
- *for every $g \in \mathcal{G}^*$, and every Left and Right option term g^L and g^R appearing in L_g and R_g, add the sentence $\forall xyz(y < x < z \rightarrow g^L(x, y, z) < g(x) < g^R(x, yz))$;*
- *the axiom scheme interpreting the ordered field structure of $Q_{\mathcal{G}}$*
- *for every $g \in \mathcal{G}^*$ such that g is monotonic/injective/onto in* No*, an axiom interpreting g as a monotonic/injective;*
- *for every $g \in \mathcal{G}^*$ such that g is a homomorphism with respect to $h, k \in \mathcal{G}^*$, i.e. if $\forall xy(g(h(x, y)) = k(g(x), g(y)))$ over the surreals, then include a corresponding sentence in $T_{\mathcal{G}}$.*

Lemma 15. *Let $\mathcal{G} \subset \mathcal{G}$ be a proper set such that $\mathsf{VR}(\mathcal{G}) = \alpha$. Then for all γ such that* No$(\varphi_{\alpha+1}(\gamma) = A_\gamma$, *we have $A_\gamma \models T_{\mathcal{G}}$.*

Proof. By the definition of Veblen rank, we have that $\mathcal{G}^*[A_\gamma] \subset A_\gamma$. In particular, we have that $\mathcal{G}^*[A_\gamma] = A_\gamma$ since we may vacuously include the identity function or multiplication by 1 or addition by 0 in $\mathcal{G}^*$. So the first two items will be satisfied, as every $A_\gamma \models$ RCF as $\varphi_{\alpha+1}(\gamma)$ will always be an epsilon number, and because the second bullet point is an axiom scheme for universal formula, we can truncate at $\varphi_{\alpha+1}(\gamma)$ and preserve the ordering of terms.

Furthermore, A_γ necessarily must *contain* $Q_{\mathcal{G}}$ for all γ by virtue of being closed under $\mathcal{G}^*$, and so each A_γ will also necessarily interpret the ordered field structure of $Q_{\mathcal{G}}$.

Finally, for the final two bullet points, each sentence is either universal or inductive. If universal with respect to No, this follows immediately given the closure under $\mathcal{G}^*$. It remains then to check that injections and surjections are satisfied.

However, this also follows by closure under $\mathcal{G}^*$: if $\mathsf{g} \in \mathcal{G}^*$ is a global injection, its restriction must also be an injection. □

Corollary 14. *Given proper subset $\mathcal{G}$, we have that $\mathsf{T}_{\mathcal{G}} \models$ ordered $\mathsf{Q}_{\mathcal{G}}$-vector space.*

Proof. This follows immediately from the fact that for every field extension L/K, we may regarded L as a K-vector space, and from the fact that every $\mathcal{M} \models \mathsf{T}_{\mathcal{G}}$ contains an isomorphic copy of $\mathsf{Q}_{\mathcal{G}}$, so we may consider M a field extension of $\mathsf{Q}_{\mathcal{G}}$. □

Theorem 63. *$\mathsf{T}_{\mathcal{G}}$ is an $\forall\exists$-theory, and thus $\mathrm{No} \models \mathsf{T}_{\mathcal{G}}$*

Proof. It is immediate that $\mathsf{T}_{\mathcal{G}}$ consists of $\forall\exists$-sentences, and by Lemma 15, we have $\mathsf{A}_\gamma \models \mathsf{T}_{\mathcal{G}}$ for every γ. Furthermore, it is immediate that

$$\bigcup_{\gamma \in \mathrm{On}} \mathsf{A}_\gamma = \mathrm{No},$$

from which we can conclude that $\mathrm{No} \models \mathsf{T}_{\mathcal{G}}$.

□

Proposition 33. $\mathsf{VR}(\mathcal{G}^*) = \mathsf{VR}(\mathsf{T}_{\mathcal{G}})$.

Proof. This follows by unfolding the definition of Veblen rank on $\mathsf{VR}(\mathcal{G}^*)$ and $\mathsf{VR}(\mathsf{T}_{\mathcal{G}})$.

First we note that $\mathsf{VR}(\mathcal{G}^*) \geq \mathsf{VR}(\mathsf{T}_{\mathcal{G}})$, since the Veblen rank of the theory is the supremum of the Veblen rank of the sentences in $\mathsf{T}_{\mathcal{G}}$, and since $\mathsf{T}_{\mathcal{G}}$ consists of sentences in $\mathcal{L}$ (and not $\mathcal{L}_{\mathsf{A}_0}$), the supremum of the Veblen rank of sentences in $\mathsf{T}_{\mathcal{G}}$ is the supremum of the terms appearing in the sentences of $\mathsf{T}_{\mathcal{G}}$. Since every term appearing in $\mathsf{T}_{\mathcal{G}}$ is an element of $\mathcal{G}^*$, the inequality immediately follows.

On the other hand, suppose that $\mathsf{VR}(\mathsf{T}_{\mathcal{G}}) < \mathsf{VR}(\mathcal{G}^*)$. Then there is some $\mathsf{t} \in \mathcal{G}^*$ such that t does not appear in any sentence in $\mathsf{T}_{\mathcal{G}}$.If t fails to appear in any sentence of $\mathsf{T}_{\mathcal{G}}$, this means that in the base model A_0 of $\mathsf{T}_{\mathcal{G}}$, we have

$$\mathsf{A}_0 \not\models \forall \bar{x}\, (\mathsf{t}(\bar{x}) = 0 \vee \mathsf{t}(\bar{x}) > 0 \vee \mathsf{t}(\bar{x}) < 0)\,.$$

However, this is absurd by trichotomy since each t is an entire genetic function, so it is guaranteed to takes some surreal value, and if it is not everywhere zero, then it necessarily is either greater than or less than zero for some point in A_0, since A_0 is closed under all terms $\mathsf{t} \in \mathcal{G}^*$. Thus $\mathsf{VR}(\mathsf{T}_{\mathcal{G}}) \geq \mathsf{VR}(\mathcal{G}^*)$, whence $\mathsf{VR}(\mathsf{T}^*_{\mathcal{G}}) = \mathsf{VR}(\mathcal{G}^*)$. □

Lemma 16. *For quantifier-free $\mathcal{G}^*$-definable classes E, $\chi_{\mathsf{E}} \in \mathcal{G}^*$. Furthermore, if $\mathsf{VR}(\mathcal{G}^*) = \alpha$, then $\mathsf{E}_0 \subseteq \mathsf{A}_0$, where $\mathsf{E}_0 = \{x \in \mathsf{E} \colon \iota x < \varphi_{\alpha+1}(0)\}$. In particular, if E is bounded, then whenever* Off $< \inf \mathsf{E}$, $\inf \mathsf{E} = \inf \mathsf{E}_0 \in \mathsf{A}_0$ *and similarly for* $\sup \mathsf{E} <$ On, $\sup \mathsf{E} = \sup \mathsf{E}_0 \in \mathsf{A}_0$*, and if an isolated point $\mathsf{p} \in \mathsf{E}$, then $\mathsf{p} \in \mathsf{E}_0$.*

Proof. If E is a $\mathcal{G}^*$ definable class, then there is a term $\mathsf{t} \in \mathcal{G}^*$ such $x \in \mathsf{E}$ if and only if $\mathsf{t}(x) = 0$. Following the discussion of primitive characteristic functions in Chapter 6.2, we may be able to define χ_E outright if it is a finite collection of points and intervals. Otherwise, we may define the characteristic function with respect to the term t as:

$$\chi_E = \left\{ \frac{-t(x)^2}{1-(t(x))^2} \right\} \mid \{\} .$$

In both cases, it follows that $\chi_E \in \mathcal{G}^*$.

The furthermore comment follows directly, since we're restricting from branches in our definable class to branches in the initial subtree A_0, which will contain all branches of height below $\varphi_{\alpha+1}(0)$. Finally, if E is bounded, then bounds must appear in E_0 since A_0 is the least initial subtree closed under all terms in $\mathcal{G}^*$. □

CHAPTER 8

CONCLUSION AND FUTURE RESEARCH

In the course of writing this dissertation, several interesting topics were developed, but set aside in order to focus on providing a solid groundwork for future investigations into theories with genetic functions. Although we have established a complexity bound on arbitrary families of genetic functions, this dissertation only hints at several potential directions for future research for the characteristic zero case, nor does it have several of the stronger results that were initially pursued. We hope to remedy this situation with the following section, which consists of conjectures and some additional exposition for several outstanding open problems related to the material of this dissertation.

The first conjecture we consider concerns a necessary property for a continuous genetic function:

Conjecture 1. *If a genetic function* g *is continuous, that is, if* $\forall \varepsilon > 0 \exists \delta > 0, |x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$*, then there is some* $\gamma \in \textsc{On}$ *such that for all* $\delta \geq \gamma$*,* $\mathsf{VR}(g, \delta) = 0$*.*

The motivation for this conjecture is that certain well-behaved genetic functions, such as polynomials, may have Veblen rank greater than 0, but otherwise tend towards 0 as we approach arbitrarily large truncated trees.

One potential work around for this is to simply redefine Veblen rank as the lim inf of the partial Veblen ranks. The drawback to such an approach is that the information contained with respect to partial Veblen ranks actually gives a finer understanding of the sequence of truncated subtrees that satisfies a given theory, as opposed to the ones that eventually satisfy a theory. This is particularly important when considering extensions with respect to a given subtrees, as we can inductively build off the complexity of known branches.

We similarly conjecture that:

Conjecture 2. *Given $\mathcal{G}$, a set of genetic functions, if $T_{\mathcal{G}}$ is o-minimal, then there is some $\gamma \in \mathrm{On}$ such that for all $\delta \geq \gamma$, $\mathsf{VR}(g, \delta) = 0$.*

As a counter-example for the reverse direction of this conjecture we note that there are cases of Veblen rank 0 functions that are discontinuous everywhere, such as ω. In the particular case of ω, certain 1-variable definable classes are not able to be described as a finite union of intervals and points, namely, the class of fixed points of ω. It is of great interest to this author to provide necessary and sufficient conditions for a generating set of genetic functions to yield an o-minimal theory.

Perhaps an easier conjecture left unproven due to time constraints is the following:

Conjecture 3. *Every entire genetic function has the sup property as described in [4].*

Specifically, we say $f : \mathrm{No}^{n+1} \to \mathrm{No}$ has the **sup property** if and only if for all $\bar{d} \in \mathrm{No}^n$ and for all $a, b, c \in \mathrm{No}$ such that $a < b$, the infimum and supremum of the following classes:

$$\{x \in \mathrm{No} \colon a < x < b < \wedge f(x, \bar{d}) \leq c\}$$

$$\{x \in \mathrm{No} \colon a < x < b < \wedge f(x, \bar{d}) \geq c\}$$

are in $\mathrm{No} \cup \{\mathrm{Off}, \mathrm{On}\}$.

This ought to follow directly from the fact that by Fornasiero's definition, $\mathfrak{a} > \mathrm{Off}$ and $\mathfrak{b} < \mathrm{On}$, and for each of these classes, when we restrict an entire genetic function f to the interval $(\mathfrak{a}, \mathfrak{b})$, the infinima and suprema should be at least $\mathfrak{a}$ and at most $\mathfrak{b}$ respectively, as f has no singularities.

The following is another conjecture that should follow straightforwardly from the definitions herein:

Conjecture 4. *For every entire genetic function* $\mathfrak{g}$ *and every* $\delta \ni \gamma$ *and* $\bar{\mathsf{e}} \in \mathrm{No}^{|\mathfrak{g}|+2}$*,* $\mathsf{S}_\gamma(\mathsf{X}[\bar{\mathsf{e}}]) \sqsubset \mathsf{S}_\delta(\mathsf{X}[\bar{\mathsf{e}}])$*.*

Additionally, we anticipate that there may be a rather simple classification structure of the possible weakenings of $\leq_s$ along our generalized notion of nested truncation rank. A description of some constraints on entire genetic functions g so that the corresponding nested truncation rank is non-trivial, as well as a possible statement of the criterion by which we can organize and understand the consequences of these weakenings would be helpful. In line with that, we conjecture the following:

Conjecture 5. *The only non-trivial simplicity preserving functions* f *, in that sense that* f *is simplicity preserving if and only if* $\mathfrak{a} <_s \mathfrak{b}$*, then* $\mathsf{f}(\mathfrak{a}) <_s \mathsf{f}(\mathfrak{b})$ *for all* $\mathfrak{a}, \mathfrak{b}$*, are the Veblen functions.*

We also presently lack constraints that can be placed on $\mathcal{G}$ so that we may define $\Gamma_{\mathcal{G}}$ as monomials expressed in terms of $\mathfrak{g} \in \mathcal{G}$. Some constraints would be helpful so that we can meaningfully expand $\mathcal{L}_{\mathsf{or}}$ by $\mathcal{G}$ and interpret models of our corresponding $\mathcal{L}$-structures as Hahn series which interpret the diagram implies by the options of the genetic functions of $\mathcal{G}$.

In particular, we wish to find constraints on $\mathcal{G}$ so that models of $\mathsf{T}_{\mathcal{G}}$ have natural interpretations as Hahn series. This leads to the following conjecture:

Conjecture 6. $\mathsf{A} \models \mathsf{T}_{\mathcal{G}}$ *is isomorphic to an initial substructure of* No *if and only if* A *is isomorphic to a cross-sectional, truncation closed subfield of a* $\mathcal{G}$*-structured Hahn field.*

We also would like to see if similar notions of genetic function can be defined for the characteristic p case by using the generalization of minimal excludent described in [22]. Similarly, we are interested in finding a recursive construction of the p-adic fields that corresponds to the natural tree structure. Indeed, we suspect that such constructions naturally can be found by carefully amending the definition of multiplication when isolating possible ring structures inside PG.

Finally, while in the course of writing this dissertation the author believed to have found a proof that all genetic functions correspondingly defined Jonsson theories when taking the inductive fragment of theory of the minimal truncated tree corresponding to the Veblen rank of g, the actual details of the proof failed in a general setting precisely because there was no clear way to establish what we anticipate is the equivalent condition of being truncation closed and cross-sectional. Indeed, further work needs to be done spelling out the connections between the model theory of valued fields and the various proposed structures above. We are motivated to answer these questions in part due to a desire to express how come the surreal numbers can be endowed with genetic functions that allow No to be the absolutely saturated model for well-known model complete theories. We weaken this to an absolutely homogeneous models for a general setting, but having some criterion for establishing when extending $\mathcal{L}_{\mathsf{or}}$ by a set of genetic functions and RCF by a corresponding set of sentences capturing the properties of $\mathcal{G}$ leads to a Jonsson or model complete theory respectively seems just out of reach. Hence our motivation to answer the conjectures made above before venturing any further in this direction.

While there are indeed many potential future directions, we find the above to be sufficient first steps for future research in applying the results of this dissertation.

CHAPTER 9

ADDENDUMS

9.1 Additional Results for Reduction

Reduction is intimately connected to the standard Krull valuation of a surreal number. We will let $\ell : \mathrm{No}^{\times} \to \mathrm{No}$ denote the leader function, defined by

$$a = \sum_{\nu a} \omega^{a_i} r_i \mapsto a_0$$

9.1.0.1 Passing Theorems

The following theorems pertain to reduction being determined in part, but not entirely, by the power tower of the leading monomial term of any given surreal number.

Theorem 64. *For every surreal number* a*, we can iterate* ℓ *only finitely many times before we terminate at a real number or an epsilon number.*

Proof. First, since $\ell : \mathrm{No}^{\times} \to \mathrm{No}$, any sequence formed by iterating ℓ will terminate whenever the leading term has a degree 0 monomial term. This occurs precisely whenever we have a real number.

Note that otherwise, if the leading exponent of a is a tower of infinite height, i.e. that we can iterate ℓ infinitely many times without termination, then the induced sequence of exponents formed by iterating ℓ must stabilize at an epsilon number.Specifically, infinite towers only occurs whenever the leading term is a eventually a fixed point of the ω map, which by definition must be an epsilon number. □

We then have the following four theorems that handle passing between surreal numbers and their leaders.

Theorem 65 (Passing Theorem 1). *If* $a, b \in \mathrm{No}_{>0}$ *such that* $\ell(a) \in \mathbb{N}$*, then* $\ell(a) \frown \ominus \sqsubseteq \ell(b) \wedge r_0 \in (0,1) \iff b \multimap_1 a$.

Proof. In the forward direction, suppose that $b_0 \in \mathbb{R}^\times$ and $a_0 \in \mathbb{N}$ such that $a_0 \frown \ominus \sqsubseteq b_0$, and without loss of generality, we have that $r_0 \in (0,1)$, so that $b = \omega^{b_0} s_0 + \text{o.t.}$ and $\omega^{a_0} r_0 + \text{o.t.}$. Then we have

$$(a) = \langle \omega^{a_0}, 0 \rangle \frown \langle \omega^{a_0} \alpha_0(r_0^\flat), \omega^{a_0} \beta_0(r_0^\flat) \rangle \frown \mathrm{op} = \langle \omega^{a_0}, \omega^{a_0} \beta_0(a_0) \rangle \frown \mathrm{o.p.}$$

$$(b) = \langle \omega^{a_0}, \omega^{a_0+1} \beta_0(b_0) \rangle \frown \text{o.p.}$$

since $0 < r_0 < 1$ entails that $\alpha_0(r_0^\flat) = 0$. Thus $b \multimap_1 a$.

Conversely, supposing the hypothesis of the theorem is satisfied and $b \multimap_1 a$, and $s_0, r_0 > 0$, then

$$\alpha_0(a) = \alpha_0(b) = \omega^{\alpha_0(a_0)}(\alpha(r_0^\flat) + 1) = \omega^{\alpha(a_0)}.$$

Thus $\alpha_0(r_0^\flat) = 0$, and therefore $r_0 \in (0,1)$. □

Theorem 66 (Passing Theorem 2). *Suppose that* $a, b \in \text{No}\backslash\mathbb{R}$ *such that the ordinal heads of* a *and* b *are trivial, i.e.* $\omega^{a_0}r_0, \omega^{b_0}s_0 \notin \text{On}$. *Further suppose that* $r_0 \in \mathbb{R}\backslash\mathbb{D}$. *Then* $b \multimap_1 a$ *if and only if* $(\ell(b), s_0) \multimap (\ell(a), r_0)$ *such that if* $(\ell(b), s_0) \multimap_2 (\ell(a), r_0)$, *then* $r_0 \in (0,1)\backslash\mathbb{D}$.

Proof. In the forward direction, first suppose that $b \multimap_1 a$. Then $\alpha_0(a) = \alpha_0(b)$ and $1 \leq \beta_0(a) \leq \beta_0(b)$, while considering the sign series expansion of a and b relative to their Conway normal form we see:

$$
\begin{aligned}
(a) &= (\sum_{\nu a} \omega^{a_i} r_i) \\
&= \langle \omega^{\alpha_0(a_0)}, \omega^{\alpha_0(a_0)+1}\beta_0(a_0)\rangle \frown \cdots \frown_{\phi r_0} \langle \omega^{a_0^+}\alpha_0(r_i^\flat), \omega^{a_0^+}\beta_0(r_i^\flat)\rangle \frown \cdots \\
&= \langle \alpha_0(a), \beta_0(b)\rangle \frown \cdots
\end{aligned}
$$

$$
\begin{aligned}
(b) &= (\sum_{\nu b} \omega^{b_i} s_i) \\
&= \langle \omega^{\alpha_0(b_0)}, \omega^{\alpha_0(b_0)+1}\beta_0(b_0)\rangle \frown \cdots \frown_{\phi s_0} \langle \omega^{b_0^+}\alpha_0(s_i^\flat), \omega^{b_i^+}\beta_0(b_0^\flat)\rangle \frown \cdots \\
&= \langle \alpha_0(b), \beta_0(b)\rangle \frown \cdots
\end{aligned}
$$

In the case where $\ell(a) \in \text{On}$, by our initial hypothesis that $\omega^{a_0}r_0 \notin \text{On}$, the sign sequence of a becomes:

$$(a) = \langle \omega^{\alpha_0(a_0)}, 0\rangle \frown_{\phi r_0} \langle \omega^{a_0^+}\alpha_0(r_0^\flat), \omega^{a_0^+}\beta_0(r_0^\flat)\rangle \frown \cdots$$

so we must have $r_0 \in (0,1)$, as

$$\omega^{a_0}\alpha_0(r_0) = \omega^{\alpha_0(b_0)} = \omega^{a_0}.$$

On the otherhand, if $\ell(a) \notin \mathrm{On}$, it still follows that $\omega^{\alpha_0(a_0)} = \omega^{\alpha_0(b_0)}$ and $1 \leq \omega^{\alpha_0(a_0)+1}\beta_0(a_0) \leq \omega^{\alpha_0(b_0)+1}\beta_0(b_0)$, whence $\alpha_0(a_0) = \alpha_0(b_0)$ and $1 \leq \beta_0(a_0) \leq \beta_0(b_0)$, and thus $b_0 \multimap_1 a_0$.

In the converse direction, first suppose that $\ell(b) \multimap_1 \ell(a)$. Then by our hypotheses, $\alpha_0(\ell(a)) = \alpha_0(\ell(b))$ and $1 \leq \beta_0(\ell(a)) \leq \beta_0(\ell(b))$. Since $a = \omega^{a_0}r_0 + \text{o.t.}$ and $b = \omega^{b_0}s_0 + \text{o.t.}$ where $a_0 = \ell(a), b_0 = \ell(b)$, we must have the following sequences:

$$(a) = \langle \omega^{\alpha_0(a_0)}, \omega^{\alpha_0(a_0)+1}\beta_0(a_0)\rangle \frown \text{o.p.}$$

$$(b) = \langle \omega^{\alpha_0(b_0)}, \omega^{\alpha_0(b_0)+1}\beta_0(b_0)\rangle \frown \text{o.p},$$

and therefore we have $\alpha_0(a) = \alpha_0(b)$ and thus $1 \leq \beta_0(a) \leq \beta_0(b)$, whence $b \multimap_1 a$.

Now suppose instead that $(\ell(b), s_0) \multimap_2 (\ell(a), r_0)$, i.e. $\ell(a) \in \mathrm{On}$ and $r_0 \in \mathbb{R}\backslash\mathbb{D}$ such that $\ell(a) \frown \ominus \sqsubseteq \ell(b)$, and further that $r_0 \in (0,1)\backslash\mathbb{D}$. Then the first pair of the sign expansion of $a = \omega^{a_0}r_0 + \text{o.t.}$ is given by

$$\langle \omega^{a_0}, \omega^{a_0}\beta_0(r_0)\rangle,$$

while the first pair of the sign expansion of b is given by

$$\langle \omega^{a_0}, \omega^{a_0+1}\beta_0(b_0)\rangle,$$

whence we can conclude that $b \multimap_1 a$.

□

Theorem 67 (Passing Theorem 3). *If* $a, b \in \mathrm{No}^\times$ *such that* $a > b$ *and* $(a_0, r_0) \in \mathrm{No}\backslash\mathrm{On} \times \mathbb{R}^\times \cup \mathrm{On} \times [-1, 1]\backslash\mathbb{D}$ *and* $(b_0, s_0) \in \mathrm{No}\backslash\mathrm{On} \times \mathbb{R}^\times$*, then*

$$b \multimap_1 a \iff (b_0, s_0) \multimap (a_0, r_0)$$

Proof. The converse direction is immediate by examination on the first pair of a and b.

In the forward direction, suppose that $a > b$ such that for the leading monomials of a and b, we have $(a_0, r_0) \in (\mathrm{No}^\times \times \mathbb{R}^\times)\backslash(\mathrm{On} \times \mathbb{D})$ and $(b_0, s_0) \in \mathrm{No}\backslash\mathrm{On} \times \mathbb{R}^\times$. Further suppose that $b \multimap_1 a$.

Then there exists some $x \in \mathrm{On}$ such that for all $y \leq x$, we have $b(x) = \ominus \wedge b(y) = a(y)$.

Then $\alpha_0(a) = \alpha_0(b)$ and $1 \leq \beta_0(a) \leq \beta_0(b)$, and in particular since the leading terms of a are $\omega^{a_0} r_0$ and b are $\omega^{b_0} s_0$, we have

$$\alpha_0(a) = \alpha_0(\omega^{a_0} r_0) = \alpha_0(\omega^{b_0} s_0) = \alpha_0(b)$$

$$1 \leq \beta_0(\omega^{a_0} r_0) = \beta_0(a) \leq \beta_0(b) = \beta_0(\omega^{b_0} s_0).$$

Now without loss of generality, suppose that $a > b > 0$. Then since $b_0 \in \text{No}\backslash\text{On}$, we have

$$(\omega^{b_0} s_0) = (\omega^{b_0}) \frown \langle \omega^{b_0^+} \alpha_0(s_0^\flat), \omega^{b_0^+} \beta_0(s_0^\flat) \rangle \frown \text{l.t.} = (\omega^{b_0}) \frown \text{l.t.}$$

$$(\omega^{a_0} r_0) = (\omega^{a_0}) \frown \langle \omega^{a_0^+} \alpha_0(r_0^\flat), \omega^{a_0^+} \beta_0(r_0^\flat) \rangle \frown \text{l.t.}$$

If $a_0 \in \text{No}\backslash\text{On}$, then we pass our analysis solely to (ω^{a_0}). In this case, we find that

$$\alpha_0(\omega^{a_0} r_0) = \alpha_0(\omega^{a_0}) = \omega^{\alpha_0(a_0)} = \alpha_0(\omega^{b_0}) = \omega^{\alpha_0(b_0)}$$

and thus $\alpha_0(a_0) = \alpha_0(b_0)$. Hence

$$1 \leq \beta_0(a) = \omega^{\alpha_0(a_0)+1} \beta_0(a_0) \leq \beta_0(b) = \omega^{\alpha_0(b_0)+1} \beta_0(b_0),$$

whence

$$1 \leq \beta_0(a_0) \leq \beta_0(b_0),$$

and therefore, we find that $b_0 \multimap_1 a_0$.

If $a_0 \in \text{On}$, then $r_0 \in [-1, 1]\backslash\mathbb{D}$, and since

$$\alpha_0(a) = \omega^{a_0} = \alpha_0(b) = \omega^{\alpha_0(b)}$$

$$1 \leq \beta_0(a) = \omega^{a_0} \beta_0(r_0) < \omega^{a_0+1} \beta_0(s_0) = \beta_0(b)$$

we see that $\mathfrak{a}_0 \frown \ominus \sqsubseteq \mathfrak{b}_0$, and thus we have $(\mathfrak{b}_0, \mathfrak{s}_0) \multimap_2 (\mathfrak{a}_0, \mathfrak{r}_0)$, whence $(\mathfrak{b}_0, \mathfrak{s}_0) \multimap (\mathfrak{a}_0, \mathfrak{r}_0)$. □

Collecting the first three passing theorems together, the following is immediate.

Lemma 17. *Suppose that* $\mathfrak{a}, \mathfrak{b} \in \mathrm{No}$ *such that* $\mathfrak{b} \multimap_1 \mathfrak{a}$. *Then* $(\ell(\mathfrak{b}), \mathfrak{s}_0) \not\multimap (\ell(\mathfrak{a}), \mathfrak{r}_0)$ *if and only if*

- *Both* $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}$*; or*
- $\ell(\mathfrak{a}) \in \mathrm{On}$ *and* $\mathfrak{r}_0 \in \mathbb{D} \cup (-\infty, -1) \cup (1, \infty)$*; or*
- $\ell(\mathfrak{b}) \in \mathrm{On}$.

Theorem 68 (Passing Theorem 4)**.** *Let* $\mathfrak{a}, \mathfrak{b} \in \mathrm{No}$ *and* $\mathfrak{r}, \mathfrak{s} \in \mathbb{R}^{\times}$ *such that* $(\mathfrak{b}, \mathfrak{s}) \multimap (\mathfrak{a}, \mathfrak{r})$. *Then there is a finite sequence* $(\mathfrak{b}_{0,i}; \mathfrak{a}_{0,i})$ *of maximal length* n *such that for each* $i < n$, $\mathfrak{b}_{0,i} \multimap_1 \mathfrak{a}_{0,i}$.

Proof. The finiteness condition follows by Theorem 64, as we pass to the leaders of a and b. Then by Theorems 28 and 64, we know that the chain will terminate either once condition 2 is satisfied, the leaders are epsilon numbers, or the final leaders are real numbers. Then we can generalize the proofs of Theorems 66 and 65 to obtain our desired result. Finally, we note that $(\mathfrak{b}_{0,n}, \mathfrak{s}_{0,n}) \not\multimap (\mathfrak{a}_{0,n}, \mathfrak{r}_{0,n})$ may happen under the conditions appearing in Lemma 17. □

9.1.0.2 Intervals of Reduction

In order to prove a general product lemma, for a given pair $\mathfrak{a} > \mathfrak{b}$ such that $\mathfrak{b}$ is non-trivially reduced with respect to a, it will be essential to characterize the intervals to which $\mathfrak{c}$ can belong such that translating by $\mathfrak{c}$ will either preserve reduction, as well as where translating by $\mathfrak{c}$ result in no reduction occurs.

Theorem 69 (Interval of Reduction). *1. Given* $a \in \text{No}\backslash\text{On}$*, if*

(a) $a > 0$*, then* $b \multimap_1 a$ *if and only if* $b \in \bigcap_{\alpha \in \alpha_0(a)} (\alpha, a)$

(b) $a < 0$*, then* $b < a \iff b \multimap_1 a$

2. Given $a \in \text{On}$*, and* $r \in \mathbb{R}\backslash\mathbb{D}$ *and* $s \in \mathbb{R}^\times$*, then* $b \in \bigcap_{\alpha \in a} (\alpha, a) \iff (b, s) \multimap (a, r)$

Proof. For 1.a, suppose that $b \multimap_1 a$. Then $b < a$ and $\alpha_0(a) = \alpha_0(b)$. Thus $\alpha < b$ for all $\alpha \in \alpha_0(b)$ by the lexicographical ordering $<$ on No, whence

$$b \in \bigcap_{\alpha \in \alpha_0(b)} (\alpha, a) = \bigcap_{\alpha \in \alpha_0(a)} (\alpha, a)$$

as desired.

On the other hand, suppose that $b \in \bigcap_{\alpha \in \alpha_0(a)} (\alpha, a)$. Then $b < a$, and $\alpha_0(b) > \alpha$ for all $\alpha \in \alpha_0(a)$. Since $b < a \leq \alpha_0(a)$, it follows that $\alpha_0(b) = \alpha_0(a)$ and further, that $\beta_0(b) \geq \beta_0(a)$. But then $b \multimap_1 a$.

For 1.b, this is immediate, as $b < a < 0$ entails that $\alpha_0(a) = \alpha_0(b) = 0$ and $\beta_0(a) \leq \beta_0(b)$.

For item 2, we use the same argument as 1.a and our hypothesis that $r \in \mathbb{R}\backslash\mathbb{D}$ to reach our conclusion. □

We have two immediate corollaries by the application of Theorem 69.

Corollary 15. *Suppose that* $a \in \text{No}_{>0}\backslash\text{On}$ *and* $c \in \text{No}$ *such that* $b + c \in \text{No}\backslash\text{On}$ *and* $a + c \in \text{No}\backslash\text{On}$*. Finally suppose that* $b \multimap_1 a$*. Then* $b + c \multimap_1 a + c \iff b + c \in \bigcap_{\alpha \in \alpha_0(a+c)} (\alpha, a + c)$.

Corollary 16. *Given* $\mathfrak{a}, \mathfrak{b} \in \text{No}\backslash\text{On}$ *such that* $\mathfrak{b} \multimap_1 \mathfrak{a}$ *and* $\mathfrak{c} \in \text{No}$*, then* $\mathfrak{b} + \mathfrak{c} \not\multimap_1 \mathfrak{a} + \mathfrak{c}$ *if and only if either*

1. $\mathfrak{a} + \mathfrak{c} \in \text{On}$*; or*

2. $\mathfrak{b} + \mathfrak{c} \in \text{On}$*; or*

3. $\mathfrak{b} + \mathfrak{c} \notin \bigcap_{\alpha \in \alpha_0(\mathfrak{a}+\mathfrak{c})} (\alpha, \mathfrak{a} + \mathfrak{c})$.

Using these results, we can further characterize the intervals in which $\mathfrak{c}$ depending on whether adding c causes reduction to occur.

Theorem 70. *Suppose that* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} > 0$ *and* $\mathfrak{b} \multimap_1 \mathfrak{a}$*. Then the following are equivalent:*

1. $\mathfrak{b} + \mathfrak{c} \not\multimap_1 \mathfrak{a} + \mathfrak{c}$*;*

2. $\exists \alpha' \in \alpha_0(\mathfrak{a} + \mathfrak{c}).(\mathfrak{b} + \mathfrak{c} \leq \alpha')$*;*

3. $\mathfrak{a}_{n_\mathfrak{a}} = \mathfrak{b}_{n_\mathfrak{b}} = \mathfrak{c}_{n_\mathfrak{c}} \wedge t_{n_\mathfrak{c}} > -r_{n_\mathfrak{a}} \wedge (t_{n_\mathfrak{c}} > -s_{n_\mathfrak{b}} \rightarrow (\mathfrak{a}_{n_\mathfrak{a}} \in \text{On} \wedge \lceil t_{n_\mathfrak{c}} + r_{n_\mathfrak{a}} \rceil > \lceil t_{n_\mathfrak{c}} + s_{n_\mathfrak{b}} \rceil))$.

Proof. First, since $\mathfrak{b} \multimap_1 \mathfrak{a}$, and $\mathfrak{a} > \mathfrak{b} > 0$ and $\mathfrak{c} > 0$ by hypothesis, we have that $\alpha_0(\mathfrak{a}) = \alpha_0(\mathfrak{b}) > 0$ and $\alpha_0(\mathfrak{a} + \mathfrak{c}), \alpha_0(\mathfrak{b} + \mathfrak{c}) > 0$. Furthermore, we note that $\mathfrak{b} \multimap_1 \mathfrak{a}$ entails that $\mathfrak{a}, \mathfrak{b} \in \text{No}\backslash\text{On}$, and that $\mathfrak{b} + \mathfrak{c} \not\multimap_1 \mathfrak{a} + \mathfrak{c}$ entails that $\alpha_0(\mathfrak{b} + \mathfrak{c}) \in \alpha_0(\mathfrak{a} + \mathfrak{c})$. From here, we proceed to prove the equivalences as follows:

(1) ⇒ (2) By the contrapositive, if for all $\alpha' \in \alpha_0(\mathfrak{a} + \mathfrak{c})$ we have $\mathfrak{b} + \mathfrak{c} > \alpha'$, then we have $\mathfrak{b} + \mathfrak{c} \in \bigcap_{\alpha_0(\mathfrak{a}+\mathfrak{c})} (\alpha', \mathfrak{a} + \mathfrak{c})$, so $\mathfrak{b} + \mathfrak{c} \multimap_1 \mathfrak{a} + \mathfrak{c}$;

(2) ⇒ (1) If there is some $\alpha' \in \alpha_0(\mathfrak{a} + \mathfrak{c})$ such that $\mathfrak{b} + \mathfrak{c} \leq \alpha'$. Let α' be the least such α' where this holds. Then $\alpha' = \alpha_0(\mathfrak{b} + \mathfrak{c})$. Since $\alpha_0(\mathfrak{b} + \mathfrak{c}) < \alpha_0(\mathfrak{a} + \mathfrak{c})$, we have that $\mathfrak{b} + \mathfrak{c} \not\multimap \mathfrak{a} + \mathfrak{c}$.

(3) ⇒ (2) Further supposing that $a_{n_a} = b_{n_b} = c_{n_c} \wedge t_{n_c} \in (-r_{n_a}, -s_{n_b}]$, then since $\alpha_0(a) = \alpha_0(b)$ by our initial hypothesis, we find that

$$\begin{aligned}
\alpha_0(b+c) &= \sum_{i<n_a} \omega^{a_i} r_i + \sum_{k<n_c} \omega^{c_k} t_k + \alpha_0(\omega^{a_{n_c}}(s_{n_c} + t_{n_c})) \\
&= \sum_{i<n_a} \omega^{a_i} r_i + \sum_{k<n_c} \omega^{c_k} t_k + 0 \\
&< \sum_{i<n_a} \omega^{a_i} r_i + \sum_{k<n_c} \omega^{c_k} t_k + \alpha_0(\omega^{a_{n_c}}(r_{n_c} + t_{n_c})) \\
&= \alpha_0(a+c)
\end{aligned}$$

Similarly, if $t_{n_c} > s_{n_b}$, then we have the implication being satisfied that $\alpha_0(t_{n_c} + r_{n_a}) > \alpha_0(t_{n_c} + s_{n_b})$, whence $\alpha_0(a+c) > \alpha_0(b+c)$.

(2) ⇒ (3) By contraposition, suppose that $a_{n_a} = b_{n_b} = c_{n_c}$ implies $t_{n_c} \notin (-r_{n_a}, -s_{n_b}]$ and $t_{n_c} > -s_{n_b}$ implies that either $a_{n_a} \notin \text{ON}$ or $\alpha_0(t_{n_c} + r_{n_a}) = \alpha_0(t_{n_c} + s_{n_b})$.

If we have $a_{n_a} = b_{n_b} = c_{n_c}$ and $t_{n_c} \notin (-r_{n_a}, -s_{n_c}]$, then either $t_{n_c} + s_{n_b} < t_{n_c} + r_{n_a} < 0$, and thus $\alpha_0(b+c) = \alpha_0(a+c)$, whence for all $\alpha' \in \alpha_0(a+c)$, $\alpha' < b+c$; or $t_{n_c} > s_{n_b}$. If $t_{n_c} > s_{n_b}$ and $a_{n_a} \in \text{No}\backslash\text{On}$, then it is immediate that since $\alpha_0(a_{n_a}) = \alpha_0(b_{n_b}) = \alpha_0(c_{n_c})$, we have $\alpha_0(b+c) = \alpha_0(a+c)$. If $a_{n_a} \in \text{On}$, $\alpha_0(b+c) = \alpha_0(a+c)$ still follows from the hypothesised implication that $\alpha_0(t_{n_c} + r_{n_c}) = \alpha_0(t_{n_c} + r_{n_c})$.

□

By similar reasoning, we have the following for reduction of the second type.

Corollary 17. *Suppose that* $a, b, c > 0$ *and* $(b, s) \multimap_2 (a, r)$*. Then the following are equivalent:*

1. $b + c \not\multimap_1 a + c$;

2. $\exists \alpha' \in \alpha_0(a + c).(b + c \leq \alpha')$;

3. $b_{n_b} = c_{n_c} \wedge (a_{n_a-1} \neq b_{n_b} \rightarrow t_{n_c} \in (0, -s_{n_b}]) \wedge (a_{n_a-1} = b_{n_b} = c_{n_c} \rightarrow t_{n_c} \in (-r_{n_a-1}, -s_{n_b}])$.

Proof. It will suffice to show that (3) ⇒ (2) and ¬(3) ⇒ ¬(2). For (3) ⇒ (2), suppose that (3) holds and $b_{n_b} \neq a_{n_a-1}$. Then because $a \frown \ominus \sqsubseteq b$, the n_b^{th} summand of b must be negative, so in particular $s_{n_b} < 0$ and we must have $\beta_0(b) < \omega^{a_{n_a-1}}$. If $t_{n_c} > 0$ such that $s_{n_b} + t_{n_c} < 0$, then by Theorem 43, we will have $\alpha_0(b + c) < \alpha_0(a + c)$. On the other hand, if $a_{n_a-1} = b_{n_b} = c_{n_c}$, since $a \frown \ominus \sqsubseteq b$, the sign of the $b_{n_b}^{th}$ term must agree with the final summand of a; so we must have $s_{n_b} > 0$ and thus we must have $\beta_0(b) \geq \omega^{a_{n_a-1}}$. So if we have $t_{n_c} \in (-r_{n_a-1}, -s_{n_b}]$, we find that $\alpha_0(b + c) < \alpha_0(a + c)$ since $t_{n_c} + r_{n_a-1} > 0 \geq t_{n_c} + s_{n_b}$.

For the contrapositive form of the forward direction, we may suppose that $b_{n_b} = c_{n_c}$.

In the case that $a_{n_a-1} \neq s_{n_b}$ and $t_{n_c} \notin (0, -s_{n_b})$, by the reasoning above, we must have that $s_{n_b} > 0$. Given $t_{n_c} \notin (0, -s_{n_b})$, then either $t_{n_c} < 0$ or $t_{n_b} + s_{n_b} > 0$, so in either case by Theorem 43, we will have $\alpha_0(b + c) = \alpha_0(b) + \alpha_0(c)$, and $1 \leq \beta_0(a + c) \leq \beta_0(b + c)$.

Finally, if $a_{n_a-1} = b_{n_b}$, by the reasoning above, $s_{n_b} > 0$, and if $t_{n_c} \notin (-r_{n_a}, -s_{n_b}]$, application of Theorem 43 to the two cases where either $t_{n_c} \leq -r_{n_a}$ or$t_{n_c} > -s_{n_b}$. In either case, $\alpha_0(b+c) = \alpha_0(b) + \alpha_0(c)$ and $1 \leq \beta_0(a + c) \leq \beta_0(b + c)$. □

Corollary 18. *If* $b \multimap_1 a$ *and* $a > b > 0$*, then the following are equivalent:*

1. $b + c \multimap_1 a + c$

2. $\forall \alpha' \in \alpha_0(a+c).(\alpha' < b+c)$

3. $a_{n_a} = b_{n_b} = c_{n_c} \to t_{n_c} \notin (-r_{n_a}, -s_{n_b}] \wedge t_{n_c} > -s_{n_b} \to \lceil t_{n_c} + r_{n_a} \rceil = \lceil t_{n_c} + r_{n_b} \rceil$.

Theorem 71. *Suppose that* $a > b > 0 > c$ *such that* $b \multimap_1 a$. *Then* $b + c \not\multimap a + c$ *if and only if either*

1. $c \in [-a, -b]$ *or*

2. $c \in (-b, 0)$ *and*

   (a) $|c|$ *has a non-trivial ordinal head if and only if* $\forall i \in n_{|c|} \exists j \in n_a$ *such that* $c_i = a_j = b_j$ *and* $t_i + s_j \geq 0$ *and*

   (b) $b_{n_b} = c_{n_{|c|}}$ *and*

   (c) *There is a* $\mu \in \nu a \cap \nu b \backslash n_a$ *such that for all* $n_a \leq i \leq \mu$, $a_i = b_i = c_i$, *for all* $n_a - 1 < i < \mu$, $t_i + s_i = t_i + r_i = 0$ *and* $\lceil t_\mu + r_\mu \rceil > \lceil t_\mu + s_\mu \rceil$.

Proof. First, in the converse direction, in case 1, where $c \in (-a, -b)$, we have $\alpha_0(b+c) = 0$ and $\alpha_0(a +_c) > 0$, so we have $b + c \not\multimap a + c$ immediately. In the second case, where $c \in (-b, 0)$, and c satisfies conditions (a) to (c), since a non-trivial ordinal head remains an ordinal by (a), by (b) if $r_{n_a} \neq s_{n_b}$, we find that the sign value changes for $b + c$, and thus $b + c \not\multimap a + c$. Finally, if $r_{n_a} = s_{n_b}$, then by (c), we have that the sign value also changes.

In the forward direction, we prove this by contraposition, in which case, it will suffice to show that for the case where $c \in (-b, 0)$ and $(a), (b)$ and $\neg(c)$ hold, then $b + c \multimap_1 a + c$, as we find that $0 > a + c > b + c$ will hold for all $c < -a$. In this case, without loss of generality, suppose that the ordinal head of $|c|$ agrees with the ordinal heads of a and b, and that the $n_a = n_b = n_{|c|}$

terms annihilate one another. Finally, after supposing that $t_{n_{|c|}} + s_{n_b} = 0 = t_{n_{|c|}} + r_{n_a}$, we have $\lceil t_\mu + s_\mu \rceil = \lceil t_\mu + r_\mu \rceil$, since $a > b$. But then we have $b + c \multimap_1 a + c$ as b+c and a+c agree on their leader. □

Corollary 19. *Suppose that* $a > b > 0 > c$ *and* $r, s \in \mathbb{R}$ *such that* $(b, s) \multimap_2 (a, r)$. *Then* $b + c \not\multimap_1 a + c$ *if and only if*

1. $c \in [-a, -b]$ *or*
2. $c \in (-b, 0)$ *and*
   - $|c|$ *has a non-trivial ordinal head if and only if* $\forall i \in n_{|c|} \exists j < n_a - 1$ *such that* $c_i = a_j = b_j$ *and* $t_i + s_j \geq 0$*; and*
   - $c_{n_{|c|}} = b_{n_b} = a_{n_a - 1} \wedge t_{n_c} \in (-r_{n_a}, -s_{n_b}]$.

Theorem 72. *Suppose that* $c > 0$ *and that* $0 > a > b$. *Then* $b + c \multimap_1 a + c$ *if and only if*

1. $c \in (0, -a)$ *OR*
2. $c \in (-b, \text{On})$ *and the ordinal head of* c *remains an ordinal when being decremented by* $\beta_0(b) = \beta_0(a)$. *Precisely,*
   - $\forall i \in n_c \exists j \in n_{|a|} \exists k \in n_{|b|}.[(c_i \geq a_j) \wedge (c_i = a_j \to c_i = b_k) \wedge (c_i = a_j \to t_i + r_j \geq 0) \wedge (c_i = b_k \to t_i + s_k \geq 0)]$ *and*
   - $c_{n_c} \geq b_{n_{|b|}} \wedge ((a_{n_{|a|}} = b_{n_{|b|}} = c_{n_c}) \to sgn(t_{n_c} + s_{n_{|b|}}) = sgn(t_{n_c} + r_{n_{|a|}}))$.

Proof. First suppose that $c > 0 > a > b$. As we have for several other theorems in this dissertation, we will prove the converse and the contrapositive.

In the converse direction, supposing first that $c \in (0, -a)$. Then we have that $b+c < a+c < 0$, and so $b + c \multimap_1 a + c$ follows.

Now supposing that $c \in (-b, \text{On})$ such that the two conditions on c hold. The two conditions listed entail that $0 < b + c < a + c$ and that $\alpha_0(b + c) = \alpha_0(a + c)$ following an application of Theorem 43 and Theorem 44.

In the forward direction, if $c \in [-a, -b]$, then $a + c \geq 0 \geq b + c$ with one of the inequalities being strict, and so $b + c \not\multimap a + c$. Thus it will suffice to check that no reduction occurs when $c \in (-b, \text{On})$ in the following two cases:

1. there is some $i \in n_c$ such that for each $j \in n_{|a|}$ and each $k \in n_{|b|}$ either $c_i < a_j$ or $c_i = a_j \wedge c_i \neq b_k$ or $c_i = a_j \wedge t_i + r_j < 0$ or $c_i = b_k \wedge t_i + s_k < 0$;
2. $c_{n_c} \geq b_{n_{|b|}} \wedge a_{n_{|a|}} = b_{n_{|b|}} = c_{n_c}$ and $\text{sgn}(t_{n_c} + s_{n_{|b|}}) \neq \text{sgn}(t_{n_c} + r_{n_a})$

In both cases, application of Theorems 43 and 44 will show that $\alpha_0(a + c) \neq \alpha_0(b + c)$, and thus no reduction occurs. □

We find as immediate corollary:

Corollary 20. *Suppose that* $c > 0 > b$ *and* $r, s \in \mathbb{R}$ *such that* $(b, s) \multimap_2 (0, r)$. *Then* $b+c \not\multimap_1 c$ *if and only if* $c \in (-b, \text{On})$.

Finally,

Theorem 73. $b + c \multimap_1 a + c$ *for all negative surreal numbers* $a > b$ *and* c.

Proof. This follows immediately from condition 1 defining $\multimap$. □

One use of reduction of the first type is that we can study the behavior of the ω map. Precisely,

Lemma 18. *For* $a, b \in \mathrm{No}$ *such that* $b \multimap_1 a$, $\iota(b^{o}a) = \iota(b)$ *if and only if* $\iota(\omega^{b^{o}a}) = \iota(\omega^{b})$.

Proof. In the forward direction, suppose that $\iota(b^{o}a) = \iota(b)$ and let $d = a \cap b$. It follows that either:

1. There must be some $i \in \phi b \backslash \phi d$ such that $\alpha_i \geq \iota(b \restriction i)$ or $\beta_i \geq \iota(b \restriction i)$, i.e. there is some pair of signs after the shared head between a and b such that one of the two ordinals is of a length that absorbs the length of d in b, and thus of $b^{o}a$ since the ordinals of this pair are untouched by reduction;
2. there is $i \in \phi b \backslash \phi d$ such that $\sum_{j<i} \alpha_j + \beta_j \leq i$.

Applying the sign sequence theorem, we see that

$$\iota(\omega(b^{o}a)) = \langle \omega^{d^+}, \omega^{d^+ + 1}\beta_{\phi d}\rangle \frown_{\phi d < i < \phi b} \langle \omega^{\gamma_i(b)}, \omega^{\gamma_i(b)+1}\beta_i(b)\rangle$$

$$\iota(\omega(b) = \langle \omega^{\gamma_0(b)}, \omega^{\gamma_0(b)+1}\beta_0(b)\rangle \frown \cdots \frown_{\phi d < i < \phi b} \langle \omega^{\gamma_i(b)}, \omega^{\gamma_i(b)+1}\beta_i(b)\rangle$$

But by the absorbing α_i or β_i, or the boundedness of the first i many summands, the lengths of $\iota(\omega^{b^{o}a})$ and $\iota(\omega^{b})$ will both entirely determined by their tails after the ϕd pair.

In the converse direction, suppose that $\iota(\omega(b^{o}a)) = \iota(\omega(b))$. Setting $d = a \cap b$, we note that necessarily $\alpha_i(b) = \alpha_i(d)$ for all $i \in \phi d$ and $\beta_i(b) = \beta_i(d)$ for all $i \in \phi d \backslash \{\sup \phi d\}$ since $b \leq d$, so let b' denote the surreal number obtained by restricting b to the first ϕd pairs and deleting the first $\beta_{\sup\phi d}(d)$ from $\beta_{\sup\phi d}(b)$. Note that if ϕd is a limit, then we may omit b' entirely. Since $d \sqsubseteq b$, we note that $d^+ := \gamma_{\phi d}(b)$ if ϕd is a limit, and otherwise $d^+ := \gamma_{\sup\phi d}(b)$. Thus we have

$$(b^o a) = \langle d^+, \beta_{\sup\phi d}(b')\rangle \frown_{\phi d \leq i < \phi b} \langle \alpha_i(b), \beta_i(b)\rangle$$

$$(b) = \frown_{i \in \phi b} \langle \alpha_0(b), \beta_i(b)\rangle$$

while

$$\iota\omega(b^o a) = \omega^{d^+} \oplus \omega^{d^+ + 1}\beta_{\sup\phi d}(b') \oplus \bigoplus_{\phi d \leq \mu < \phi b} (\omega^{\gamma_\mu(b)} \oplus \omega^{\gamma_\mu(b)\oplus 1}\beta_\mu(b))$$

$$\iota\omega(b) = \bigoplus_{\mu < \phi b} (\omega^{\gamma_\mu(b)} \oplus \omega^{\gamma_\mu(b)\oplus 1}\beta_\mu(b))$$

So if

$$\iota(\omega^{b^o a}) = \omega^{d^+} \oplus \omega^{d^+ + 1}\beta_{\sup\phi d}(b') \bigoplus_{\phi d \leq \mu < \phi b} \omega^{\gamma_\mu(b)} \oplus \omega^{\gamma_\mu(b)\oplus 1}\beta_\mu(b) = \iota(\omega^b),$$

It follows that the length of $\bigoplus_{i<\phi d} (\omega^{\gamma_i(b)} \oplus \omega^{\gamma_i(b)\oplus 1}\beta_i(b))$ is absorbed into the ordinal determined by the tail starting with the ϕd pair. But then it follows that $\iota(b^o a) = \iota(b)$, as no $\beta_i(b)$ for $i \in \phi d$ can be greater than b^+. □

By trichotomy, we have an immediate corollary,

Corollary 21. $\iota(\omega(b^o a)) < \iota(\omega(b))$ *if and only if* $\iota(b^o a) < \iota(b)$.

9.2 Model Complete Theories of the Surreal Numbers with Genetic Function

There is no first order axiomatization of the surreal tree since fullness and completeness are not first order properties. Instead these results should be handled as follows: first examine theories that are model complete or admit QE, and show how these structures are all embeddable as initial subtrees of surreal. Further, show that if A is an initial subtree of the surreals that satisfies a first order theory T satisfied by the surreals consisting of the order relation and genetic function symbols, then A is an elementary substructure. This latter result is the actual result we want. The author first noticed that the following results from [17] corresponded to well-known model complete theories:

Theorem 74. *Every divisible ordered abelian group is isomorphic to a recursively defined initial subgroup of* No*.*

Theorem 75. *Every real-closed ordered field is isomorphic to a recursively defined initial subfield of* No*.*

In turn, these proofs made use of the following results:

Theorem 76. *If* A *is a divisible initial subgroup of* No*, and* $\mathfrak{a}$ *is the simplest element of* No *that fills a cut in* A*, then the divisible subgroup of* No *generated by* $\mathsf{A} \cup \{\mathfrak{a}\}$ *is an initial subgroup of* No*.*

and

Theorem 77. *IF* A *is a real-closed initial subfield of* No *and* $\mathfrak{a}$ *is the simplest element of* No *that fills a cut in* A*, then the real-closed subfield of* No *generated by* $\mathsf{A} \cup \{\mathfrak{a}\}$ *is an initial subfield of* No*.*

From here, the author wondered, in line with Ehrlich's result that the surreal numbers are the unique homogeneous[1] universal[2] ordered field by virtue of the *s-hierarchical structure* defining the surreal numbers if this same structure could be used to explicitly demonstrate model-completeness of the theories whose models are universally embedding into the surreals. If so, then there is a general programme for constructing model-complete theories with linearly ordered universes, as a sufficient condition for model completeness is for T to be given by $\mathcal{L} = \{<\} \cup \mathcal{F} \cup \mathcal{C}$ where $<$ is interpreted to be a linear order, $\mathcal{F}$ consists of genetic functions, and for all models of T to be isomorphic to initial subtrees of No.

In [17], the author proves the following equivalence

Theorem 78. *For a lexicographically ordered binary tree* $\langle \mathsf{A}, <, <_{\mathsf{s}} \rangle$*, the following are equivalent:*

1. $\langle \mathsf{A}, <_{\mathsf{s}} \rangle$ *is a full lexicographically ordered binary tree;*

2. $\langle \mathsf{A}, <, <_{\mathsf{s}} \rangle$ *is a complete lexicographically ordered binary tree;*

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[1] Homogeneous in the sense that every isomorphism of subfields of an ordered field A can be extended to an automorphism of A.

[2] An ordered field A is universal if every ordered field whose universe is a Class of NBG can be embedded in A.

3. *the intersection of every nested sequence* I_α *for* $0 \le \alpha \in \beta \in \mathrm{On}$ *of nonempty convex subclasses of* $\langle A, <, <_s \rangle$ *is nonempty and contains a simplest member.*

While these results are not expressible with first-order formula in $\mathcal{L} = \langle <, <_s \rangle$, nonetheless following Ehrlich [13], we have that No is the absolutely universal homogeneous model for inductive theories $T_\mathcal{G}$ written in the signature of $\{0, 1, +, \times, <\} \cup \mathcal{G}$, where $\mathcal{G}$ consists of *genetically* defined function symbols. Further, as discussed in Chapter **??**, we conjectured that in certain cases where the Veblen rank is zero, then $T_\mathcal{G}$ would be a model-complete theory.

The following subsections explore several familiar examples of model complete theories T over linearly ordered structures. In each, we establish the following:

1. How $\mathcal{L}$ consists of $<$ and genetic functions;
2. How each $\mathcal{M} \models T$ can be realized as an initial subtree of No;
3. How $\mathcal{M}$ being an initial subtree of No entails that $\mathcal{M}$ is an *(elementary) substructure* of No.

Addressing the first item requires proving that each function symbol is genetically defined, i.e. it is recursively defined using cuts and possesses the uniformity property. This is the most important step in that we need to establish that when each function symbol f is genetically defined, we also have that No $\models T$, where T is the first order theory given with these function symbols.

The second item consists of demonstrating that No is the universal embedding object for the theory T. In effect, this amounts to demonstrating that for each $A \models T$, we can find some truncation $\mathrm{No}(\alpha) = \{a \in \mathrm{No} \mid \iota(a) < \alpha\}$ closed under the composition of the various function and relation symbols in $\mathcal{L}$ such that α is the least ordinal such that we can build a 1-1 model isomorphism between an initial subtree $B \subset \mathrm{No}(\alpha)$ and A. However, with demonstrating that $\mathbb{Q}$ and $\mathbb{R}$ are initial subtrees isomorphic to some A will be unnecessary - in these specific cases, both $\mathbb{Q}$ and $\mathbb{R}$ are initial subtrees of No by virtue of the fact that they're both subsets of the Archimedean Class $[1]_{\sim}$. Specifically, for any non-dyadic rational number r, the sets $L_r, R_r \subseteq [1]_{\sim}$, so we have that there is immediate agreements with respect to the predecessors of $\mathbb{Q}$ and $\mathbb{R}$ respectively in No. In general, we'll be making use of this sort of argument with respect to the Archimedean Class decomposition of A, and find that in some cases, the only models where the agreement is not on the nose between truncated trees and models A will be $\mathbb{Q}$ and $\mathbb{R}$.

The final item amounts to recasting the approaches used in [36] and [37] to prove the model completeness, namely showing that for $A \subseteq \mathrm{No}$ such that $A, \mathrm{No} \models T$, and A is an initial subtree, then A is *existentially closed*. This in turn consists of: (1) establishing that any finite set of equations involving the genetic functions of $\mathcal{L}$ and parameters in A is solvable in A provided it is solvable in No; (2) solvability over parameters drawn from $A \subset \mathrm{No}(\alpha)$ in No entails solvability in any $\mathrm{No}(\gamma) \models T$ where $\gamma \geq \alpha$. A final bit of work will need to be done to address cases where there are multiple $A_1, A_2, \ldots$ such that each $A_i \subset \mathrm{No}(\alpha)$ and $A_i, \mathrm{No}(\alpha) \models T$, but

$A_i \neq A_j$ are pairwise distinct.[1]

The following subsections run through these three steps to establish this general pattern.

9.2.0.1 Dense Linear Orders

Let $\mathcal{L} = \langle < \rangle$, and let DLO be the theory of dense linear orders without endpoints. There are no genetic functions that we need to interpret.

We have that No $\models$ DLO by interpreting $<$ as the lexicographical ordering on No. Now let $\mathcal{M} \models$ DLO. To show that $\mathcal{M}$ is isomorphic to an initial subtree of No, one can use a back-and-forth argument as follows:

Letting A be the underlying universe of $\mathcal{M}$, set $\alpha := \mathsf{o.t.}A$, and let $(a_i : i \in \alpha)$ be a listing of the elements of M. Set β to be the least ordinal such that $\alpha \leq \mathsf{o.t}({}^{<\beta}2)$, and let $B := \mathrm{No}(\beta)$. Further, let

$$(b_j : j \in \mathsf{o.t.}({}^{<\beta}2) \wedge \forall j \forall x \in \mathsf{pr}_B(b_j) \exists i < j . x = b_j)$$

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[1] The canonical example of a phenomenon like this would be $\mathbb{Q}, \mathbb{R}$ and $\mathrm{No}(\omega^2)$ and $T \models$ DLO, where DLO is the theory of dense linear orders without endpoints. $\mathbb{Q}$ and $\mathbb{R}$ both consists of branches of length $\leq \omega$, the height of $\mathbb{Q}$ and $\mathbb{R}$ are $\omega + 1$, but $\mathrm{No}(\omega + 1)$ fails to be DLO since $-\omega, \omega$ are endpoints. As mentioned in the previous paragraph, we can sidestep the question about whether $\mathbb{Q}$ and $\mathbb{R}$ are initial subtrees because we know they're both contained in the same Archimedean Class $[1]_\sim$.

be a listing of the branches of B such that for each b_j, every predecessor b_i of b_j satisfies $i < j$.[1]

We only need to build an embedding of A At the initial stage, let $A_0 = B_0 = f_0 = \emptyset$.

At stage $n+1 = 2m+1$, ensure $a_m \in A_{n+1}$ as in the standard back-and-forth construction, i.e. if $a_m \in A_n$, set $A_{n+1} = A_n$, $B_{n+1} = B_n$, and $f_{n+1} = f_n$, and if $a_m \notin A_n$, find $b \in B \backslash B_n$ which is in the image under f_n of the cut of a_m in A_n, i.e. we'll add a_m to the domain of the partial embedding by finding some $b \in B \backslash B_n$ such that $x < a_m \iff f_n(x) < b$ for all $x \in A_n$.

At stage $n+1 = 2m+2$, we need $b_m \in B_{n+1}$. If $b_m \in B_n$, fix $A_{n+1} = A_n$, $B_{n+1} = B_n$ and $f_{n+1} = f_n$. Otherwise, find $a \in A$ such that the image of the cut of a in A_n is the cut of b_m in B_n. In particular, every b_m has been chosen so that each of its predecessors appears in B_n, so each odd stage will ensure that canonical cuts are preserved under these partial embeddings. From this, we have the final image $f(A)$ realizes an initial subtree in B.

Finally, for stages where n is a limit ordinal, let $A_n := \bigcup_{m<n} A_m$, $B_n := \bigcup_{m<n} B_m$, and $f_n := \bigcup_n f_m$. This will suffice to ensure that each A_n is isomorphic to an initial subtree of B, and that $A = \bigcup A_n$ itself is isomorphic to an initial subtree of B.

Finally, it is straightforward to check that this is existentially closed. Suppose that $\phi(x; \bar{y})$ is a quantifier-free $\mathcal{L}$ formula and $\bar{a} \in A$. Let $\psi(\bar{a}) = \exists x \phi(x; \bar{a})$. Since each $\phi(x; \bar{a})$ can be decomposed into a disjunctive normal form where each conjunct containing x and some a_i

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[1] There are multiple ways this can be accomplished. One way is to add the dyadic rationals by tree rank, and then add the branches of length ω once we've reached the ω stage. If we're filling in the reals, we only need to continue to list branches of length ω. If we're filling in any other dense linear order, we set for each limit ordinal stage γ a new branch b^* of height ω, and then for each successor stage $\gamma + n$, we concatenate b^* with the branch of the dyadic rationals corresponding to n. Once we've filled out all the branches of height ω that appear in B, we move on to $\omega 2$, and repeat this process transfinitely until we've filled in all branches.

in $\bar{a}$ corresponds to $a_i < x$, $a_i = x$, or $x < a_i$ by trichotomy, if $\text{No} \models \psi(\bar{a})$, then there is some $b \in \text{No}$ satisfying at least one of the non-contradictory conjuncts of ψ when written in disjunctive normal form. In particular, if $b = a_i$ is one of the conjuncts, we're done. Otherwise, we may let b be the simplest element satisfying the cut given by a_i. In particular, this b MUST be in a, since this cut consists of only finitely many clauses consisting of elements in A, and $A \models \text{DLO}$. Thus we have $\text{No} \models \psi(\bar{a}) \iff A \models \psi(\bar{a})$. Furthermore, by the same simplicity argument, we find that existential closure follows for all intermediate models $\text{No}(\alpha) \supseteq A$, and since every dense linear ordering can be embedded into a minimal $\text{No}(\alpha)$, we find existential closure follows in general for any $A \subset A'$ by the same appeal to simplicity.

9.2.0.2 Ordered Divisible Abelian Groups

First, let $\mathcal{L} = \langle +, -, <, 0 \rangle$. We give genetic definitions to $+$ and $-$ as follows:

$$a + b := \left\{ a^L + b, a + b^L \right\} \mid \left\{ a^R + b, a + b^R \right\}$$

$$-a := \left\{ -(a^R) \right\} \mid \left\{ -(a^L) \right\}$$

Proofs of the uniformity property for addition can be found in [1], whence $+$ is a genetic function. Proof that the additive inverse map has the uniformity property follows by a routine induction argument - in this case, we note that we can establish the uniformity property directly by taking any cofinal set F with respect to L_a and coinitial set G with respect to R_a and by induction showing that $-G$ becomes cofinal with $(-a)^L$ and $-F$ becomes coinitial with $(-a)^R$. Finally, the interpretation of 0 in No is immediate.

Secondly, as established by Theorems 7-9 of [17], every ordered vector space over an Archimedean ordered field is isomorphic to an initial subtree of No, and consequently, every ordered divisible abelian group is isomorphic to an initial subtree of the surreal numbers. Additionally, we establish below in Theorem ?? that $\mathrm{No}(\alpha) \models \mathrm{ODAG}$ if and only if $\alpha \in \Delta''\mathrm{On}$.

Finally, we establish existential closure in the same manner as in the case of dense linear orders. Explicitly, given $A \models \mathrm{ODAG}$ such that $A \subseteq \mathrm{No}$, we know that A is an initial subtree, and that we may assume without loss of generality that for any quantifier free $\mathcal{L}$ formula $\phi(x, \bar{a})$ is written in disjunctive normal form. We may further assume by trichotomy, that $\phi(x, \bar{a})$ consists of conjunctions of atomic formula of one of the three following forms[1]:

(L) $nx < t_i(\bar{a})$;

(E) $nx = t_i(\bar{a})$;

(R) $t_i(\bar{a}) < nx$.

where $n \in \mathbb{Z}$.

Finally, if any of the atomic formula are of the form (E), then by the closure of A under addition, if $\mathrm{No} \models \exists\phi(x, \bar{a})$, then the witness to $nx = t_i$ must be necessarily be in A. So we may further assume that $\phi(x, \bar{a})$ consists solely of atomic formula of form (L) or (R). But then $\phi(x, \bar{a})$ corresponds to a cut consisting of finitely many clauses. In particular, since $+, -$ are genetic functions, and A is an initial subtree closed under $+, -$, any cut whose clauses consist of terms

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[1]Any relation of the form $tR(x + a)$ is equivalent to $(t + (-a))Rx$ by the axioms of ordered divisible abelian groups.

expressed in $\mathcal{L}_A$ must be satisfied by some element in A as there are only finitely many clauses in the cut.

Thus, if No $\models \exists x\phi(x, \bar{a})$, then for $b := \{t_i^L\} \mid \{t_j^R\}$ where t_i are the terms appearing in the atomic formula of form (L) and t_j are the terms appearing in the atomic formula of form (R), we have No $\models \phi(b, \bar{a})$. But by the preceding argument, we have by simplicity and closure of A under $+, -$, that $b \in A$, whence $A \models \phi(b, \bar{a})$, so that $A \models \exists x\phi(x, \bar{a})$.

In fact, given that every ordered divisible abelian group is isomorphic to some initial subtree of No, for any $A \subseteq A'$ such that $A, A' \models$ ODAG, the same simplicity argument applies, whence existential closure follows.

9.2.0.3 Real Closed Ordered Fields

First, given $\mathcal{L} = \langle +, -, \cdot, 0, 1, < \rangle$, where $+, -$ are defined as before, and as we have done earlier in this dissertation, we define multiplication by

$$a \cdot b := \{a^L \cdot b + a \cdot b^L - a^L \cdot b^L, a^R \cdot b + a \cdot b^R - a^R \cdot b^R\} \mid$$

$$\{a^L \cdot b + a \cdot b^R - a^L \cdot b^R, a^R \cdot b + a \cdot b^L - a^R \cdot b^L\}.$$

The proof that $\cdot$ has the uniformity property can be found in [1]. Furthermore, multiplicative inverses are defined in [1] for all non-zero surreal numbers by way of genetic functions in Chapter 3 Chapter C of [1].

Theorem 19 of [17] combines Theorem 18 and Proposition 7 of [17] to show that every real closed field is isomorphic to an initial subtree of No. Specifically, Theorem 18 shows that an ordered field is isomorphic to an initial subfield of No if and only if it is isomorphic to a truncation complete, cross sectional subfield of a powerseries field $\mathbb{R}(G)_{\text{On}}$ where G is isomorphic to an initial subgroup of No, while Proposition 7 shows that every real-closed ordered field is isomorphic to a truncation complete, cross sectional subfield of a power series field $\mathbb{R}(G)_{\text{On}}$ where G is a divisible ordered abelian group.[1]

Finally, we can prove model completeness by adjusting the standard argument that RCF admits quantifier elimination, in $\mathcal{L}_{\mathsf{or}}$, the language of ordered rings, that can be found elsewhere in the literature (such as [38] or [39]), as summarized below.

First, recall that every quantifier free formula can be put into a disjunctive normal form such that each atomic formula is equivalent to polynomial expression $f(\nu) = 0$ or $f(\nu) > 0$ (since $f(\nu) < 0$ is equivalent to $-f(\nu) > 0$); in particular, it will suffice to examine quantifier free formula of the form $\phi(\nu, \bar{a}) := \bigwedge_{i=1}^{m} f_i(\bar{\nu}) \wedge \bigwedge_{j=1}^{m} g_j(\bar{\nu}) > 0$, where $f_1, \ldots, f_m, g_1, \ldots, g_n \in A[x]$. If $b \in \text{No}$ such that $\text{No} \models \phi(b, \bar{a})$, then b will be algebraic over A. Since $A, \text{No} \models \text{RCF}$, it follows that $b \in A^{\text{rcl}} = A$, from which we determine that it will actually suffice to consider quantifier free formula $\phi(\nu, \bar{a}) := \bigwedge_{j=1}^{m} g_j(\bar{\nu}) > 0$.

By real closure, we can factor each g_j into linear or quadratic factors of the form $(\nu - a)$ and $(\nu - a)^2 + b^2$ where $a, b \in A$ and $b \neq 0$. Furthermore, since A^2 is the positive cone of A, it

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[1]Specifically, with multiplication defined á la Hahn, $\mathbb{R}(G)_{\text{On}}$ is a field of power series of the form $\sum_{\alpha\in\beta} r_\alpha t^{y_\alpha}$ where $(y_\alpha)_{\alpha\in\beta}$ is a descending sequence of elements of an ordered abelian group G and $r_\alpha \neq 0$. A **truncation** of $\sum_{\alpha\in\beta} r_\alpha t^{y_\alpha}$ is a power series $\sum_{\alpha\in\sigma} r_\alpha t^{y_\alpha}$ where $\sigma \leq \beta$. A subfield $F \subset \mathbb{R}(G)_{\text{On}}$ is **truncation complete** whenever every truncation of a member of a subfield is in the subfield. Further, a subfield F is **cross-sectional** if $\{t^g : g \in G\} \subseteq F$.

follows that $(v-a)^2+b^2 \geq 0$ for all $a, b, v \in A$. Thus for each $g_j(v) > 0$, it follows that an even number of the linear terms must be negative, and the remaining linear terms must be positive, so without loss of generality, suppose that we have factored

$$g_j(v) = (v - a_{1_j}) \cdots (v - a_{k_j}) \cdot ((v - a_{k_j+1})^2 + b_{k_j+1}^2)) \cdots ((v - a_{n_j})^2 + b_{n_j}^2)$$

such that $a_{1_j} \leq a_{2_j} \leq \cdots \leq a_{k_j}$. Whenever k_j is even, the product of the linear terms will be positive for v lying in the intervals $(a_{k_j}, \infty), (a_{k_j-2}, a_{k_j-1}), \ldots$ or $(-\infty, a_{1_j})$, and similarly, if k_j is odd, for the intervals $(a_{k_j}, \infty), (a_{k_j-2}, a_{k_j-1}), \ldots$ or (a_{1_j}, a_{2_j}), whence $g_j(v) > 0$ is equivalent to

$$\psi_j(v) := v < a_{1_j} \vee \bigvee_{i=1}^{\frac{n_j-2}{2}} (a_{2i_j} < v < a_{(2i+1)_j} \vee a_{n_j} < v$$

whenever n_j is even or

$$\psi_j(v) := \bigvee_{i=1}^{\frac{n_j-1}{2}} (a_{(2i-1)_j} < v < a_{(2i)_j} \vee a_{n_j} < v$$

whenever n_j is odd. In either case, over $\mathcal{L}_A$, $\phi(v, \bar{a}) \leftrightarrow \bigwedge g_j(v) > 0 \leftrightarrow \bigwedge \psi_j(v)$. If we set $c = \max\{a_{n_j} \mid 1 \leq j \leq n\} + 1$, it follows that $c \in \mathrm{No}$, and furthermore, by simplicity, $c \in A$. Furthermore, $\psi_j(c)$ holds in No and thus in A, from which we witness existential closure for all intermediate models.

Alternatively, we can adjust straightforwardly note that for quantifier free formula where $\phi(v, \bar{a})$ consisting of $g_j(\bar{v}) > 0$, we can define the genetic partial function $G(v) := \{0\} | \{g_1(v), \ldots, g_n(v)\}$.

If $G(A) \neq \emptyset$, we still have failure with respect to the intervals in A derived from the factorization of the polynomials g_j, and because each g_j is a polynomial with coefficients in A, any infinitesimal extension of an element in A will leave the sign unchanged. On the other hand, because of the failure for the formula to be satisfied when looking at rays $(-\infty, a_{1_j})$ or (a_{n_j}, ∞), the polynomials are monotonically decreasing on these rays, and so any infinite element $x \in \mathrm{No}$ such that $x < A$ or $A < x$, will leave the sign value unchanged. Thus $G(\mathrm{No}) = \emptyset$. From this we can determine that $A \models \exists v \phi(v, \bar{a}) \iff \mathrm{No} \models \exists v \phi(v, \bar{a})$.

9.2.0.4 Real Closed Fields with Exponentiation

First, given $\mathcal{L} = \langle +, -, \cdot, \exp, 0, 1, < \rangle$, with all the functions except exp defined as before, we will briefly review Kruskal's exp function as described in [1]. Importantly, [1] contains proofs that exp has the uniformity, and is genetic. The following result from van den Dries and Ehrlich is of model theoretic interest [6].

Theorem 79. *The surreal numbers are a model of the elementary theory of the field of real numbers with the exponential function.*

Finally, when adapting [36] proof of model completeness, we will first define $e(x) := \exp((1+x^2)^{-1})$. As the composition of genetic functions, $e(x)$ is also genetic. Furthermore, for every $n \in \omega$ and $s \subseteq n$, we let M_n^s denote the ring of functions from $\mathrm{No}^n \to \mathrm{No}$ over k, where $k \subseteq \mathrm{No}$ and $k \models T_{\exp}$, that is generated by the functions $x_i, (1 + x_i^2)^{-1}, e(x_i), \exp(x_i)$, and the elements of k as constant functions.

Following our construction of $\mathcal{G}^*$, if we take $\mathcal{G} = \{\exp, \frac{1}{1+x^2}\}$, then $M_n^s \subset \mathcal{G}^*$. Because entire genetic functions are closed under composition, addition and multiplication, every element of

M_n^s is a genetic function. Additionally, M_n^s is Noetherian for all $n \in \omega$ and $s \subseteq n$, and each element of M_n^s is a smooth function from No^n to No. Furthermore, M_n^s is closed under differentiation, so for all $f_1, \ldots, f_n \in M_n^s$, the Jacobian $J(f_1, \ldots, f_n) \in M_n^s$.

From here we still identify the underlying real closed ordered field with the initial substructure as in the case of Real Closed Ordered Fields.

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