Review of Surreal Number Construction and Forcing Constructions

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This is a fairly broad overview of the surreal numbers, some interesting inductive constructions which allow us to regard them as universal domains for various power series fields equipped with an exponential function, as well as of logarithmic-exponential series in the sense of Kulhmann [?] and the transseries in the sense of Schmelling [?]. Furthermore, I provide some broad overview of the notion of set theoretic forcing. In the process, I hope to provide the groundwork of results necessary to perform general ordinal analysis and relate the proofs of consistency strength to constructions one can build with respect to the surreal numbers

- Eventually I will include a brief overview of the ordinal analysis to be done.

1 Surreal Numbers

This section is divided into several subsections. The goal is to provide a relatively concise overview of the following: Gonshor's monograph [?], wherein which he showed that the surreal numbers form a real closed field with an exponential and logarithmic map that agree with the ordinary real valued maps; work by Kuhlmann and Matusinski on exponential-logarithmic classes and the κ numbers [?]; the recent work of Berarducci and Mantova [?], which defined a Hardy-type derivation map and established that the surreal numbers form a field of exponential-logarithmic transseries. Throughout this review, we will build the following descending chain of proper classes of surreal numbers which are used in exploring the model theoretic properties of the surreal numbers:

$$\mathrm{No} \supsetneq \omega^{\mathrm{No}} \supsetneq \omega^{\omega^{\mathrm{No}}} \supsetneq \kappa^{\mathrm{No}} \supsetneq \varepsilon^{\mathrm{No}}.$$

The aim going forward with these results will be to study both the model theory of the surreal numbers and use those results to study various large cardinal hypothesis and notions of forcing.

We begin this section by introducing the surreal numbers, some fundamental existence and cofinality results essential for proving various uniformity theorems, several simplicity results, the construction of the reals, and some rudimentary analysis on the surreal numbers. The next subsection provides definitions for the field operations. After providing the definitions for the field operations, we introduce the definitions for the ω and ϵ numbers, and provide a brief summary of the κ numbers described in [**?**]. We then provide a definition for the exp map. Once we have a defintion of exp, we then give an overview and relevant results regarding two normals which can be put on the surreal numbers, along with the sign sequence lemma, which is yet another canonical name for the surreal numbers (precisely, the sign sequence names the branch of 2^{On} which *names* a surreal number). We then explore the log map in some detail, before describing the log-atomic numbers and finishing our exposition on the κ numbers. Finally, we provide an overview on the construction of the ∂_{BM} derivative.

1.1 Numbers and Games

Surreal numbers, denoted by No, relate the theory of transfinite numbers with mathematical games. Aside from their popular origins in Donald Knuth's *Surreal Numbers*, they were invented as a way to attain a theory of the real numbers by defining them as strengths of positions of certain games. Specifically, the surreal numbers are a class of games Game defined with respect to a transfinitely constructed sets of numbers No given by the following construction rules:

- If L, R are two sets of No such that $\forall x \in L \forall y \in R . x \leqslant y$, then there is a number $a = L \mid R = \{x^L\} \mid \{x^R\}$ (with x^L and x^R denoting typical elements of the (canonical) sets defining x);
- We inductively define the relation $\leqslant$ by $y \leqslant x := \neg\exists(x^R \leqslant y) \wedge \neg\exists y^L(x \leqslant y^L)$
- A game consists of two sets, F, G consisting of surreal numbers. We say II wins if $F < G$. Otherwise, the games are said to be **incomparable**, which is denoted by $F \| G$.

Alternatively, the surreal numbers can be defined as follows:

Definition. *a is a **surreal number** if it is a function from an initial segment of the ordinals* On *(including from the empty set) into the set $2 = \{-, +\}$.*

We define a linear ordering $<$ on the surreal numbers as follows: if α is the least ordinal such that $a(\alpha) \neq b(\alpha)$, then $a < b := (a(\alpha) = -) \vee (b(\alpha) = +)$, with the convention that $- < 0 < +$.

*The **common initial segment** of two surreal numbers a and b is the number c whose length is the least α such that $a(\alpha) \neq b(\alpha)$ and for all $\beta \in \alpha$, $c(\beta) = a(\beta) = b(\beta)$. We denote that c is an initial segment of a by $c \sqsubset a$.*

For two sets $F < G$, we denote by $F|G$ the unique c of minimal length such that $F < c < G$.

As functions from ordinals α to 2, the surreal numbers form a a well-founded partial order $<_s$ on No *called the* ***simplicity relation****, with $a <_S b$ if $a \sqsubset b$ as a function.*

Example 1. *Here's a guiding example for canonical sets:*

$$\omega - 1 := \{n\} \mid \{\omega\}$$

where n ranges over ω with $n = (+_n)$ understood to be a sequence of n plusses and $\omega = (+_\omega)$ is a sequence of ω plusses.

Definition. *Given two sets $A < B$ of surreal numbers, we define the* ***associated convex class***

$$(A; B) := \{y \in \text{No} \mid A < y < B\}.$$

By construction, every $(A; B)$ has a minimal representative with respect to $<_s$. Moreover, the canonical representation of a surreal number a are the sets $F < G$ for which all members of F, G are simpler than a, and for which a is the simplest element in $(F; G)$.

Gonshor establishes the following theorems [**?**]:

Theorem 1. *(Fundamental Existence Theorem) Let F, G denote two sets of surreal numbers such that $(a \in F \wedge b \in G) \Rightarrow a < b$. Then there exists a unique c of minimal length such that $a \in F \Rightarrow a < c$ and $b \in G \Rightarrow c < b$. Furthermore, c is an initial segment of all surreal numbers d such that $F < d < G$.*

Every cut has a well-defined element as a consequence of this Fundamental existence theorem. The following theorems are essential to proving uniformity theorems for various inductively defined operations; these theorems allow for various substitutions between canonical sets, and cofinal sets that are more appropriate for describing the operation at hand.

Theorem 2. *(Representation Theorem) Given two sets of numbers $F < G$, a pair (F', G') is* ***cofinal*** *in (F, G) if*

$$\forall a \in F \exists b \in F'(a \leqslant b) \wedge (\forall a \in G \exists b \in G').(b \leqslant a)$$

Theorem 3. *(Main Cofinality Theorem) If $a = F|G$, and $F' < a < G'$, and (F', G') is cofinal in (F, G), then $a = F'|G'$.*

Corollary 1. *Suppose (F, G) and (F', G') are mutually cofinal in one another. Then $F|G = F'|G'$.*

Theorem 4. *Let $a \in$ No. Suppose that $F' := \{b \mid b < a \wedge b \sqsubset a\}$ and let $G' := \{b \mid a < b \wedge b \sqsubset a\}$. Then $a = F'|G'$.*

Theorem 5. *(The Inverse Cofinality Theorem) Let $a = \{a^L\} \mid \{a^R\}$ be the canonical representation of a and let F, G be such that $a = F|G$. Then (F', G') is cofinal in $(\{a^L\}, \{a^R\}) = A^L|A^R$.*

1.1.1 Simplicity relation and binary trees

Surreal number can be understood as a lexicographically ordered tree with a partial order relation on denoted by $<_s$ for branch predecessors. Alternatively, we may describe This relationship has been extensively studied by Ehrlich, et. al [**?**, **?**] as forming a simplicity hierarchy (or **s-hierarchy**).

Definition. *A representation of $a = F|G$ is **simple** if $a \leqslant_s y$ implies $F < y < G$*

We now recall some definitions of trees:

Definition. *A **tree** $(T, <_s)$ is a partially ordered class such that for each $x \in T$, the predecessors,*

$$pr_T(x) := \{y \in T \mid y <_s x\}$$

form a set well-ordered by $<_s$.

*The **branches** of a tree form a maximal subclass of maximal subclasses ordered by $<_s$.*

*$x, y \in T$ are **incomparable** if $x \neq y \wedge x \nleq_s y \wedge y \nleq_s x$. We denote incomparability by $x \perp y$.*

*The **tree-rank** of $x \in T$ is denoted by $\rho_T(x)$, and is defined to be the ordinal corresponding to the well-ordering of the set $\langle pr_T(x), <_s \rangle$.*

*The α **level** of a tree $T_\alpha := \{x \in T \mid \rho_T(x) = \alpha\}$. A **root** of T is a member of the zeroth level.*

*An **immediate successor** of x is any y such that $x <_s y$ and $\rho_T(y) = \rho_T(x) + 1$. For any chain of $(x_\alpha)_{\alpha \in \beta}$ in T, y is a **successor of the chain** if $x_\alpha <_s y$ and*

$$\rho_T(y) = \inf\{\gamma \in \mathrm{On} \mid \rho_T(x) = \gamma\}$$

.

*A tree T is **binary** if each member of T has at most two immediate successors, and every chain in T of limit length has at most one immediate successor.*

*A tree T is **lexicographically ordered** if for all x, y $x \perp y$ if and only if there is $z <_s x, y$ and $x < z < y$ or $y < z < x$.*

*A tree T is **full** if every member of T has immediaate successors, and every chain of T of limit length has an imediate successor. Consequently, the universe of a full tree T forms a proper class.*

*A lexicographically ordered tree T is **complete** if for any two subsets L, R of T such that $L < R$, there is an $x \in T$ such that $L < x < R$.*

Theorem 6. *Every $(T, <, <_s)$ is a lexicographically ordered binary tree is isomorphic to an initial subtree of $(\mathrm{No}, <, <_s)$. In particular, the surreal numbers are a lexicographically ordered binary tree.*

The following are some elementary facts of $<_s$ which can be found in [**?**, **?**, **?**]:

Fact 1. *Let $a, b, c, x, y \in \mathrm{No}$:*

1. *if $c <_s x \leqslant_s y$, then $c < x \iff c < y$;*

2. *if $a = F|G$, and $F \cup G$ only contain $x <_s a$, then $F|G$ is simple.*

3. *A lexicographically ordered binary tree is complete if and only if it is full if and only if it is isomorphic to $(\text{No}, <, <_s)$.*

Proposition 1. $(\text{No}, <_s^o)$ *is a separative partial order under reverse inclusion.*

Proof. It is immediate that No is partially ordered by $<_s$, and so No will also be partially ordered by the opposite $<_s^o$, with top element 0.

Now suppose $a, b \in \text{No}$ have tree rank α, β respectively and are such that $a \nleqslant_s^o b$. Then $b \not\sqsubseteq a$, and so either $a \sqsubset b$ or $a \perp b$.

If $a \sqsubset b$, then there is some $x \in \{-, +\}$ such that $a \frown x \sqsubset b$. Let $y = \neg x$ (i.e. $\neg - = +$ and $\neg + = -$), and consider $c = a \frown y$. Then $a \sqsubset c$, hence $c \leqslant_s^o a$ and $c \perp b$ as desired. If $a \perp b$, then we may take $a = c$. □

1.1.2 Real numbers

While it can be readily seen that the surreal numbers contain the dyadic rationals, $\mathbb{D}$, we use the following definition to define the real subset of the surreal numbers.

Definition. *A **real number** is a surreal number a which is either of length $\leqslant \omega$, such that if $lh(a) = \omega$, then*

$$\forall n_0 \exists n_1 \exists n_2 [n_1 \geqslant n_0 \wedge n_2 \geqslant n_0 \wedge a(n_1) = + \wedge a(n_2) = -]$$

It is worth noting that the definition of real numbers above cannot distinguish between rational and irrational real numbers, nor can we distinguish between rational and irrational elements based on the *canonical* representation of a surreal number. To check that the set of reals forms a field, one must check the closure properties under the operations defined below in the next subsection.

The following list of facts can be found in [?] Chapter 4 Section C.

Fact 2. *1. Let F, G be non-empty sets of dyadic fractions such that $F < G$, F has no maximum and G has no minimum. Then $F|G$ is a real number.*

2. *If $a = F|G$,*

$$\forall x \in F \exists a \in \mathbb{D}_{>0} \exists y (yx + r \wedge y \in F),$$

and

$$\forall x \in G \exists r \in \mathbb{D}_{>0} \exists y (y \leqslant x - r \wedge y \in G),$$

and $F' < a < G'$, and

$$\forall r \in D_{>0} \exists x \in F' \exists y \in G' (y - x \leqslant r),$$

then $a = F'|G'$.

3. *There are an infinite number of dyadic rationals between any two distinct real a and b.*

4. *If $a = F|G$ is the canonical representation of a non-dyadic real number, then for all positive dyadic rationals r, there exist $b \in F, c \in G$ such that $c - b \leqslant r$*

Theorem 7. *The subset of reals $R \subsetneq \mathrm{No}$ has the lub property.*

Proof. While any set which has no maximum in the surreal numbers will have no least upper bound in the class of the surreal numbers as a consequence of the existence of gaps (or more simply, the infinitesimal numbers), we can prove that every bounded non-empty set of real numbers has a least upper bound *within the set of reals* using the facts above.

Let H be a non-empty set of real numbers bounded above, and let G be the set of all dyadic rationals which are upper bounds of H, and let F be the complement of G in $\mathbb{D}$. Since F, G are non-empty, the facts above, F has no maximum, and if G had a minimum b, then b would be a least upper bound to H, and thus we're done.

So, further suppose that G has no minimum. Then $r = F|G$ will be a real number. We check that $r = lubH$.

First, we note that r is an upper bound to H, as otherwise, there is some $a \in H$ such that $r < a$, and then there is some dyadic rational d such that $r < d < a$. But since $d < a \in H$, $d \in F$, and therefore since $r < d$, we contradict that $F < r$.

Finally, towards another contradiction, suppose that s is an upperbound to H such that $s < r$. Let $s < d < r$ for some $d \in \mathbb{D}$. This d is also an upper bound to H, and hence $d \in G$. But $d < r < G$. Contradiction.

Thus we have that $r = F|G$ is a real number, and since H was arbitrary, we have established that the $\mathbb{R}$ has the least upper bound property. □

Notation. *By the theorem above, we may denote the subfield $R \subset \mathrm{No}$ of reals by the ordinary boldface $\mathbb{R}$.*

1.1.3 Elementary Surreal Analysis: Gaps

While surreal numbers are definable with respect to *cuts* defined with respect to sets, one can form a cut of sorts with respect to classes, known as a *gap*, for which there is no surreal number that can satisfy the gap (as a consequence, the surreal numbers form a totally disconnected space). More precisely:

Definition.

We classify gaps into two types:

1.2 Field Operations

Definition. *We define the ring operations $+, \cdot$ as follows*

$$a + b := \{a^L + b, a + b^L\} \mid \{a^R + b, a + b^R\}$$

$$a \cdot b := \{a^L \cdot b + a \cdot b^L - a^L \cdot b^L, a^R \cdot b + a \cdot b^R - a^R \cdot b^R\} \mid \{a^L \cdot b + a \cdot b^R - a^L \cdot b^R, a^R \cdot b + a \cdot b^L - a^R \cdot b^L\}$$

Theorem 8. *The surreal numbers form a commutative ordered ring with unity.*

Proof. A full proof of this can be found in Chapter 2 of [?]. □

Definition. *If $a > 0$, then the multiplicative inverse of a is defined as follows:*

Let $a = A^L|A^R$. Define $\langle a_1, a_2, \ldots a_n\rangle$ for every finite sequence where $a_i \in A^L \cup A^R \backslash \{0\}$, and let $0 = \langle\rangle$ and $\langle a_1, \ldots a_n\rangle \deg a_{n+1} = \langle a_1, \ldots, a_n, a_{n+1}\rangle$.

For arbitrary b, define $b \deg a_i$ to be the unique solution to

$$(a - a_i)b + a_i x = 1$$

By induction, each a_i will be an initial segment of a, and will have an inverse, with uniqueness following.

Then for $a \in \mathrm{No}^\times$, $a^{-1} = F|G$ where $F = \{\langle a_1, \ldots, a_n\rangle \mid$ the number of $a_i \in A^L$ is even$\}$ and $G = \{\langle a_1, \ldots, a_n\rangle \mid$ the number of $a_i \in A^L$ is odd$\}$

One can find in [?] the proof of the following:

Theorem 9. No *is a real closed field*

1.3 ω and generalized ϵ numbers

Two interesting maps related to the study of large ordinals and cardinals are the ω and ϵ maps.

Definition. *Let $a \sim b$ if and only if there exists an integer n such that $na \geqslant b \wedge nb \geqslant a$. This equivalence relation classifies numbers by their order of magnitude.*

Let $a \gg b$ if and only if for all integers n $(nb \leqslant a)$, and $a \ll b$ if and only if $b \gg a$. It follows that each $\sim$ equivalence class is convex.

We now inductively define $\omega : \mathrm{No} \to \mathrm{No}_{>0}$ as follows:

$$\omega^a = \omega(a) := \{0, r\omega(a^L) \mid r \in \mathbb{R}_{>0}\} \mid \{s\omega(a^R) \mid s \in \mathbb{R}_{>0}\}$$

The following results from [?] warrant statement and proofs, as they will establish the properties to show that the value group of the surreal numbers are the surreals themselves.

Theorem 10. *For every $a \in \mathrm{No}_{>0}$ there exists an unique x of minimal length such that $x \sim a$.*

Proof. By well-ordering, there exists an x of minimal length such that $x \sim a$. If there existed a distinct $y \sim x \sim a$, of minimal length, then with $z \sqsubset x, y$, it would follow by the convexity of $\sim$ classes that $z \sim a$. Since $lh(z) < lh(x)$, we have a contradition of the minimality of x. □

Theorem 11. *$\omega(a)$ is defined for all $a \in \mathrm{No}$, $\omega(a) > 0$, and $a < b \Rightarrow \omega(a) \ll \omega(b)$.*

Proof. As is standard in proofs on No, we prove this by induction on the length of a.

Since $a^L < a^R$, by our inductive hypothesis, $\omega(a^L) \ll \omega(a^R)$. Hence for all positive reals r, s, we have $0 < r\omega(a^L) < s\omega(a^R)$. Hence $\omega(a)$ will be defined, and since 0 is a lower element in its definition, $0 < \omega(a)$.

Suppose $a < b$ and c is a common initial segment. Then if $c = a$ or $c = b$, we're done. Otherwise, we have $\omega(a) \ll \omega(c) \ll \omega(d)$. ☐

Theorem 12. *An element $a = \omega^b$ if and only if a is the element of minimal length in its $\sim$ equivalence class.*

Proof. Unfolding the definition, we have $\omega^b = \omega(b) = \{0, r\omega(b^l)\} \mid \{s\omega(b^R)\}$. If $x \sim \omega(b)$, then $r\omega(b^L) < x < s\omega(b^R)$ since r, s are arbitrary positive reals. Thus $\omega(b)$ must be an initial segment of x, so we have $lh(\omega(b)) \leqslant lh(x)$.

Conversely, we must show every positive element is equivalent to an element of the form $\omega(b)$. Given the inequality, $b < c \Rightarrow \omega(b) \ll \omega(c)$, if an element exists, it must be unique.

We proceed by induction. Set $a = A^L|A^R$. We have $0 \in A^L$. Every element in $A^L \cup A^R$ is equivalent to an element of the form $\omega(b)$ by our induction hypothesis.

Now set $F = \{y \mid \exists x \in A^L(x \sim \omega(y))\}$ and $G = \{y \mid \exists x \in A^R(x \sim \omega(y))\}$. Suppose that $F \cap G = \varnothing$.

We claim that $F < G$. If not, then there is $x > y$ such that $x \in F$ and $y \in G$. Then $\omega(x) \sim a^L$ and \$ω(y)∼ aR, hence $x > y \Rightarrow \omega(x) \gg \omega(y) \Rightarrow a^L \gg a^R \Rightarrow \perp$. Thus $F < G$ and there is a $z = F|G$.

Let $\omega(F)$ be the complete set of representatives for the equivalence classes containing the elements of $A^L \backslash \{0\}$, and similarly for $\omega(G)$ with respect to A^R.

We now evaluate the following three cases:

Case 1 $r\omega(x) \geqslant a$ for some positive real r and some $x \in F$. Let a^L such that $a^L \sim \omega(x)$. Then $a^L \leqslant a \leqslant r\omega(x)$. But we have $a^L \sim \omega(x) \sim r\omega(x)$, and hence $a \sim \omega(x)$. Case 2 $r\omega(x) \leqslant a$ for some positive real r and some $x \in G$. Let a^R satisfy $a^R \sim \omega(x)$. Then $r\omega(x) \leqslant a \leqslant a^R$, but $r\omega(x) \sim \omega(x) \sim a^R$, and hence $a \sim \omega(x)$. Case 3 If neither case 1 nor case 2 is satisfied, then $r\omega(F) < a < s\omega(G)$. Let $a^L \neq 0$. Then there exists some $x \in F$ such that $a^L \sim \omega(x)$, and for some real r, $r\omega(x) \geqslant a$. Similarly for a^R we have some real s such that $s\omega(x) \leqslant a^R$.

Since $a = A^L|A^R$, the cofinality condition is satisfied for $\{0, r\omega(F)\} \mid \{s\omega G\}$. Hence $a = \{0, r\omega(F)\} \mid \{s\omega G\} = \omega(z)$.

The theorem now follows since $\omega(b)$ has the minimal length property. ☐

Looking ahead, with $\omega^a = \omega(a)$ the representative of minimal length in its respective **Archimedean equivalence class**, we can put each surreal number into a Conway normal form (see below), as well as put a natural valuation on each surreal number which sends each $a = \sum \omega^{a_i} r_i$ to a_0.

Definition. *Let $\omega^{\text{No}} = \omega\text{"No} = \mathfrak{M}$ denote the group of* ***generalized monomials****.*

It is immediate by construction that $\omega^{\text{No}} \subsetneq \text{No}$.

1.3.1 *εnumbers*

Definition. *We inductively define* $\varepsilon :\to_{>0}$ *as follows: Let* $\omega_1(a) = \omega(a)$ *and* $\omega_{n+1} = \omega(\omega_n(a))$.

Let $a = A^L|A^R$ *be given its canonical representation. Let*

$$\varepsilon(a) = \{\omega_n(1), \omega_n[(a^L) + 1]\} \mid \{\omega_n[\varepsilon(a^R) - 1]\}$$

where n *ranges over positive integers.*

We note that $\varepsilon(0) = \{\omega_n(1)\} \mid \{\} = lub\{\omega_n(1)\} = \varepsilon$, the first ordinary epsilon number. It is worth remarking that Gentzen showed that transfinite induction up to $\epsilon(0)$ suffices to prove the consistency of PA [**?**]. It warrants investigation how ordinal analysis can relate to various inductively defined ordinal operations on the surreal numbers. The following proof can be found on page 122 of [**?**].

Theorem 13.

1. *$\varepsilon(a)$ is defined for all $a \in \mathrm{No}$ and $\omega(\varepsilon(a)) = \varepsilon(a)$.*
2. *$\varepsilon(a)$ is a strictly increasing function and $\varepsilon(a) > \omega_n(1)$ for all a and positive integers n.*

Definition. *Let* $\mathrm{No}_{>0}^{\gg 1}$ *denote the class of positive valued Archimedean equivalences classes whose elements* $> \mathbb{R}$.

We define $\omega^{\omega^{\mathrm{No}}}$ *to be the chain of* ***fundamental monomials*** *of* No. $\omega^{\omega^{\mathrm{No}}}$ *is a proper class which is a complete set of represetatives of the comparability classes* $\mathrm{No}_{>0}^{\gg 1}$, *with each element being the one of minimal length in its respective equivalence class.*

We now have

$$\omega^{\omega^{\mathrm{No}}} \subsetneq \omega^{\mathrm{No}} \subsetneq \mathrm{No}$$

1.3.2 Higher order fixed points

Theorem 14. *Let* $f : \mathrm{No} \to \mathrm{No}$ *be a function with the following properties:*

1. *for all a, $f(a)$ is a power of ω;*
2. $a < b \Rightarrow f(a) < f(b)$
3. *There exist fixed sets C and D such that if $a = G|H$ with G containing no maximum and H no minimum, then $f(a) = [C, f(G)]|[D, f(H)]$.*

Then there exists a function g which is onto the set of all fixed points of f and whcih satisfies the above hypothesis on the sets $f_n(C)$ and $f_n(D)$, for arbitrary positive n iterations.

1.4 exp

In this subsection we provide a brief overview of the exp map detailed in [**?**].

Definition. *For each $x \in \text{No}$ and $n \in \omega$, let*

$$[x]_n := \sum_{i \leqslant n} \frac{x^n}{n!}$$

We define $\exp :\to_{>0}$ *by*

$$exp(x) := \{0, \exp(x^L)[x - x^L]_n, expx^R[x - x^R]_{2n+1}\} \mid \{\frac{expx^R}{[x^R - x]_n}, \frac{\exp x^L}{[x^L - x]_{2n+1}}\}$$

The following facts have proofs from Chapter 10 of [?] or from immediate examples.

Fact 3. *1.* exp *is a monotonic function onto* $\text{No}_{>0}$*;*

2. $\exp \upharpoonright \mathbb{R}$ *is the real exponential function.*

3. $\exp(x + y) = \exp(x)\exp(y)$ *for all* $x, y \in \text{No}$*; furthermore,* exp *is an isomorphism between ordered abelian groups* $(\text{No}, +, <)$ *and* $(\text{No}_{>0}, \cdot, <)$.

4. $\exp(x)$ *is a power of* ω *and if* $a > 0$*, then* $\exp(\omega^a)$ *has the form* $\omega(\omega(b))$.

5. exp *is not a $<_s$$-hierarchy preserving map.*

6. For $x \in \text{No}_{>0}$, $\exp(\omega(x)) = \omega(\omega(g(x)))$*, where* $g : \text{No}_{>0} \to \text{No}$ *defined by*

$$g(x) := \{c(x), g(x^L)\} \mid \{g(x^R)\},$$

with $c(x)$ *the unique number such that* $\omega(c(x)) \sim x$.

The following result from van den Dries and Ehrlich is of model theoretic interest [?].

Theorem 15. *The surreal numbers are a model of the elementary theory of the field of real numbers with the exponential function.*

1.4.1 TODO κ numbers pt 1

1.5 Normal Forms and Standard Form

Every surreal number $x \in \text{No}$ has a **Conway normal form**. Specifically, for every $x \in \text{No}$, there is an $\alpha \in \text{On}$, a sequence of real numbers $(r_i)_{i \in \alpha}$, and a descending sequence of surreal numbers $(a_i)_{i \in \alpha}$ such that

$$x = \sum_{i \in \alpha} \omega(a_i) r_i$$

where $(a_i)_{i \in \alpha}$We define **support** of a surreal number a to be the set

$$S(a) := \{y \in \text{No} \mid \exists \beta \in \alpha(y = y_\alpha \wedge r_\alpha \neq 0)\}.$$

Definition. *Let $x \in \mathrm{No}$. If we express $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}}\mathfrak{m}$, as above:*

1. *The* ***support*** *of x is the set $S(x) := \{\mathfrak{m} \in \mathfrak{M} \mid x_{\mathfrak{m}} \neq 0\}$;*
2. *The* ***terms*** *of x are the elements of the set $\{x_{\mathfrak{m}}\mathfrak{m} \mid x_{\mathfrak{m}} \neq 0\} \subset \mathbb{R}^{\times}\mathfrak{M}$;*
3. *The* ***coefficient*** *of $\mathfrak{m}$ in x is $x_{\mathfrak{m}}$;*
4. *The* ***leading monomial*** *of x is the maximal monomial in $S(x)$;*
5. *The* ***leading term*** *of x is the leading monomial multiplied by its coefficient.*
6. *Given $\mathfrak{m} \in \mathfrak{M}$, the* ***truncation*** *of x at $\mathfrak{m}$ is the number*

$$x \upharpoonright := \sum_{\mathfrak{m} < \mathfrak{n}} x_{\mathfrak{n}}\mathfrak{n}$$

.

7. *If $y \in \mathrm{No}$ is a truncation of x, we denote this by $y \trianglelefteq x$.*

Following [?], we have the following inductively defined map $\Sigma : \mathbb{R}((\mathfrak{M})) \to \mathrm{No}$:

Definition. *Let $f \in \mathbb{R}((\mathfrak{M}))$. With $f_{\mathfrak{m}} \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$, and $f \upharpoonright \mathfrak{m}$ denoting the truncation at $\mathfrak{m}$, we define:*

1. *If $S(f) = \varnothing$, then $\Sigma f := 0 \in \mathrm{No}$;*
2. *If $S(f)$ contains a smallest monomial $\mathfrak{n}$, define*

$$\Sigma f := \Sigma f \upharpoonright \mathfrak{n} + f_{\mathfrak{n}}\mathfrak{n}$$

3. *If $S(f) \neq \varnothing$ and has no smallest monomial, with q^L, q^R arbitrary dyadic rationals such that \$$q^L$<f*

{m}<q^R\$

$$\Sigma f := \{\Sigma f \upharpoonright \mathfrak{m} + q^L\mathfrak{m}\} \mid \{\Sigma f \upharpoonright \mathfrak{m} + q^R\mathfrak{m}\}$$

In particular, we recognize that

$$\mathrm{No} = \mathbb{R}((\omega^{\mathrm{No}})) = \mathbb{R}((\mathfrak{M}))$$

Given that we may regard ω^{No} as the complete system of representatives of Archimedean equivalence classes of $\mathrm{No}_{>0}$, and that we can take the map Ind from [?] that sends each surreal number to the exponent a_0 of its leading monomial in normal form, we may regard the surreal numbers as a valued field which is its own value group. Moreover, Kuhlmann showed in [?] that Ind is the natural valuation of the real closed field No, with $Ind(\omega^a) = a$ for all $a \in \mathrm{No}$.

We can further deduce that No is a **Hahn field of series** in the following sense: [?]

Theorem 16. *1. For any $a \in \mathrm{No}$, $(\omega(\omega(a))$ is the representative of minimal length in $\mathrm{No}^{\gg 1}_{>0}$, i.e.*

$$\forall x, y \in \mathrm{No}^{\gg 1}_{>0}, x \sim_{comp} y \iff (\exists n \in \omega(x^n \geqslant y \geqslant x^{1/n})$$

We set $x \sim_{comp} \frac{1}{x}$.

2. Any $a \in \mathrm{No}$ can be uniquely written as

where for any i,

$$a_i = \sum_{j \in \lambda_i} \omega^{b_{i,j}}$$

, the $(a_i)_{i\in\lambda}, (b_{i,j})_{j\in\lambda_i}$ form descending sequences of surreals, and for any i, j, we have $s_{i,j}, r_i \in \mathbb{R}^\times$, so that

$$\omega(a_i) = \prod_{j<\lambda_i} \left(\omega^{\omega(b_{i,j}}\right)^{s_{i,j}}$$

In particular, using the definition of Hahn fields from [?], we find

$$\mathrm{No} = \mathbb{R}\left(\left(\left(\omega^{\omega^{\mathrm{No}}}\right)^{\mathbb{R}}\right)\right)$$

We let $\mathbb{J} \subset \mathrm{No}$ denote the (class) non-unital ring of infinite surreal numbers. Specifically, they're the surreal numbers whose supports have infinite monomials, so

$$\mathbb{J} := \{a \mid \forall y \in S(a) \exists z > 0 (y = \omega(z))\} = \mathrm{No}^{\gg 1} \cup \{0\}$$

It follows from the constructions above that

$$\omega(\mathrm{No}) = \exp(\mathbb{J})$$

A very important substructure of the surreal numbers are the **omnific integers**.

Definition. *The* ***omnific integers*** *are the numbers of the form $x = \{x - 1\} \mid \{x + 1\}$. The class of omnific integers, denoted Oz, has the direct sum decomposition $\mathbb{J} \oplus \mathbb{Z}$.*

Remark. *We note that surreal numbers can be given a natural direct sum decomposition of*

$$\mathrm{No} = \mathbb{J} \oplus \mathbb{R} \oplus o(1)$$

where $o(1)$ denotes the class of inifinitesimal numbers.

We may also put the surreal numbers in **Ressayre normal form**, as in each $x∈ No

$$x = \sum_{i\in\beta} \exp(y_i) r_i$$

where $(y_i)_{i\in\beta}$ is a descending sequence of surreal numbers.

Definition. *Given $x \neq 0$ with Ressayre normal form $\sum_{a \in \mathbb{J}} r_a \exp(a)$, with $r_a \neq 0$ if and only if $a \in S(x)$, we define $\ell : \mathrm{No}^\times \to \mathbb{J}$ by $\ell(x) = \max\{a \in \mathbb{J} \mid r_a \neq 0\}$.*

Remark. *The map above can be regarded as the logarithm a of the largest monomial $\mathfrak{m} = \exp(a)$ appearing in the Conway normal form of x. Further, $-\ell$ defines a Krull valuation on* No*, given that*

1. $\ell(x+y) \leqslant \max\{\ell(x), \ell(y)\}$
2. $\ell(xy) = \ell(x) + \ell(y)$.

An almost immediate consequence of these two normal forms is that the surreal numbers can be understood as a valued field which is its own valued group.

The following are facts about the normal form with respect to the simplicity hierarchy (see [?] for more details):

Fact 4.

1. *For all $a, b \in \mathrm{No}$, $\omega(a) <_s \omega(b)$ if and only if $a <_s b$.*
2. *If $a <_s b$, then $\omega(x)a <_s \omega(x)b$.*
3. $\sum_{i \in \mu} \omega(y_i) r_i <_s \sum_{j \in \nu} \omega(y_j) r_j$ *whenever $\mu <_s \nu$*
4. *If μ is a limit ordinal, then with γ ranging over μ, and n ranging over ω, we have*
$$\sum_{i \in \mu} := \{\sum_{i \in \gamma} \omega(y_i) r_i + \omega(y_\gamma)(r_\gamma - \frac{1}{2^n})\} \mid \{\sum_{i \in \gamma} \omega(y_i) r_i + \omega(y_\gamma)(r_\gamma + \frac{1}{2^n})\}$$
5. *If $r \in \mathbb{R} \backslash \mathbb{D}$, or $r \in \mathbb{D} \backslash \mathbb{Z}$ and there is no y^L, then*
$$\omega(y)r = \{\omega(y)r^L\} \mid \{\omega(y)r^R\}$$
6. *For all r^L, r^R,*
$$\omega(y)r^L <_s \omega(y)r$$
and
$$\omega(y)r^R <_s \omega(y)r$$
7. *If $r \in \mathbb{D} \backslash \mathbb{Z}$, and there exist y^L, then*
$$\omega(y)r = \{\omega(y) + r^L + \omega(y^L)n\} \mid \{\omega(y)r^R - \omega(y^L)n\}$$
8. $\omega(y)r^L + \omega(y^L)n <_s \omega(y)r$.
9. $\omega(y)r^R - \omega(y^L)n < s\omega(y)r$.
10. *For all n, $\frac{1}{2^n}\omega(x^R) <_s \omega(x)$.*

Consequently, we have

Proposition 2. *If $x \trianglelefteq y$, then $x <_s y$.*

We can also see that $\trianglelefteq$ is a weakening of $<_s$ once we have our results on the sign sequence of surreal numbers.

1.5.1 Sign-sequence representation

The following results are a summary of Gonshor Chapter 8 [?], as well as some new results of Kuhlmann and Matusinski [?]. While each author has their own preferred notation for concatenation and representing the sign sequence, we have opted to use notation keeping in line with work found in Kunen [?], Jech [?], and other more set theoretically inclined authors [?].

Definition. *Recall that the surreal numbers may be regarded as (partial) functions from* $\mathrm{On} \to 2$*, so that for two surreal numbers* a, b*, we may* ***concatenate*** *them to form a third number,* $a \frown b$*. The concatenation operation respects standard results on ordinal length, i.e.*

$$lh(a \frown b) = lh(a) \oplus lh(b)$$

as can be verified by an induction argument on the lengths of numbers.

Notation. *Every surreal number* a *can be written as a transfinite concatenation of* $-$ *and* $+$

$$a = (+_{\alpha_0} -_{\beta_0} +_{\alpha_1} -_{\beta_1} \cdots),$$

with $+_\alpha$ *(respectively* $-_\beta$*) denoting a string of* $+$ *(resp. -) of length* α *(resp.* β*), and where for any* $\mu \in \mathrm{On}$*,* $\alpha_\mu, \beta_\mu \in \mathrm{On}$ *with* α_μ *possibly being 0 for* $\mu = 0$ *or* $\mu \in Lim(\mathrm{On})$*.*

For concision, we will denote by (a) the sign sequence of a*, and write out the sign sequence as the sequence of ordered pairs* $(\langle \alpha_i, \beta_i \rangle : i \in \gamma)$*.*

Definition. *Given* $a \in \mathrm{No}$*, let* a^+ *denote the total number of* $+$ *appearing in the sign sequence of* a*, so*

$$a^+ = \sum_\mu \alpha_\mu$$

as an ordinal sum.

Given $a \in \mathrm{No}_{>0}$*, define* $a^\flat$ *to be the surreal number attained by omitting the first* $+$ *sign.*

Given $a \in \mathrm{No}_{<0}$*, define* $a^\sharp$ *to be the surreal number attained by omitting the first* $-$ *sign.*

Given a surreal in normal form $a = \sum_{i \in \lambda} \omega^{a_i} r_i$*, the* ***reduced sequence*** $(a_i^o)_{i \in \lambda}$ *is attained by omitting - in the following sign sequences:*

- *given* $\gamma \in \mathrm{On}$*, if* $a_i(\gamma) = -$ *and there exists* $j < i$ *such that \$*$a_j(\delta)$*=*$a_i(\delta)$ *for all* $\delta \leqslant \gamma$*, then omit the* δ^{th} *-;*
- *if* i *is a successor,* $a_{i-1} \frown - \sqsubset a_i$ *and if* r_{i-1} *is not a dyadic rational, then omit the - after* a_{i-1} *in* a_i*.*

The following theorems provide a concise overview of the sign sequence lemma, as well as the sign sequence of generalized epsilon numbers.

Theorem 17. *Given $a = (\langle\alpha_i, \beta_i\rangle)$, for any $\mu \in \mathrm{On}$ appearing in the sign expansion of a, we set*

$$\gamma_\mu := \sum_{\lambda\leqslant\mu} \alpha_\lambda$$

Then ω^a has the sign sequence

$$\langle\omega^{\gamma_0}, \omega^{\gamma_0+1}\beta\rangle \frown (\langle\omega^{\gamma_i}, \omega^{\gamma_1+1}\beta_i\rangle)_{0<i<\mu}$$

Theorem 18. *Given a positive real r with sign sequence $(\langle\rho_i, \sigma_i\rangle)$, the sign sequence of $\omega^a r$ is*

$$(\omega^a) \frown \langle\omega^{a^+}\rho_0^\flat, \omega^{a^+}\sigma_0\rangle \frown (\langle\omega^{a^+}\rho_i, \omega^{a^+}\sigma_i\rangle)$$

with $\omega^{a^+}\rho$ and $\omega^{a^+}\sigma$ being the standard ordinal multiplication (with absorption).

If r is a negative real, we reverse the signs.

Theorem 19. *Given $a = \sum_{i<\lambda} \omega^{a_i} r_i$,*

$$(a) = \frown_{i<\lambda} \omega^{a_i^\circ}(r_i)$$

The following theorem is a combination of theorems 9.5 and 9.6 in [**?**]

Theorem 20. *1. $a = (\langle\alpha_i, \beta_i\rangle)$ is an epsilon number if and only if $\alpha_0 \neq 0$, all $\alpha_\mu \neq 0$ are ordinary epsilon numbers such that $\alpha_\mu > lub\{\alpha_\lambda \mid \lambda < \mu\}$, and β_μ is a multiple of $\omega^{\alpha_\mu\omega}$ for all $\alpha_\mu \neq 0$, and a multiple of $\omega^{\gamma_\mu\omega}$ where $\delta_\mu = \sum_{\lambda<\mu} \alpha_\mu$ for $\alpha_\mu = 0$.*

2. Let $\gamma_\mu = \sum_{\lambda\leqslant\mu} \alpha_\lambda$. Then the μ^{th} block of + in $\varepsilon(a)$ consists of e_{γ_μ} +'s and the μ^{th} block of -'s will consist of $(\varepsilon_{\gamma_\mu})^\omega\beta_\mu$ -'s.

1.5.2 A quick review of summability

Having identified surreal numbers with $\mathbb{R}((\mathfrak{M}))$, we can explore the notion of infinite sums. Namely,

Definition. *Let $(x_i)_{i\in I}$ be an indexed set of surreal numbers. We say $(x_i)_I$ is* ***summable*** *if $\bigcup_I S(x_i)$ is reverse well-ordered, and if for each $\mathfrak{m} \in \bigcup S(x_i)$ there are only finitely many $i \in I$ such that $\mathfrak{m} \in S(x_i)$.*

When $(x_i)_I$ is summable, then the ***sum***

$$y := \sum_{i\in I} x_i$$

is the unique surreal number such that:

- $S(y) \subseteq \bigcup_I S(x_i)$

- *for every* $\mathfrak{m} \in \mathfrak{M}$, $y_{\mathfrak{m}} = (\sum_{i \in I} x_i)_{\mathfrak{m}} = \sum_{i \in I} x_{i\mathfrak{m}}$

Definition. *A function* $F : \mathrm{No} \to \mathrm{No}$ *is* ***strongly linear*** *if for all* $x = \sum x_{\mathfrak{m}} \mathfrak{m}$*,*

$$F(x) = \sum x_m F(\mathfrak{m}).$$

In particular, $(x_{\mathfrak{m}} F(\mathfrak{m}))$ *is summable.*

Proposition 3. *IF* F *is a strongly linear function, then for any summable* (x_i)*, the family* $(F(x_i))$ *is summable and*

$$F(\sum x_i) = \sum F(x_i)$$

Proof. The following one line proof is from [?]:

$$F(\sum x_i) = F\left(\sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{i \in I} x_i\right)_{\mathfrak{m}} \mathfrak{m}\right) = \sum_{\mathfrak{m} \in \mathfrak{M}} \sum_{i \in I} x_{i\mathfrak{m}} F(\mathfrak{m}) = \sum_{i \in I} F(x_i)$$

□

1.5.3 Nested truncation and standard forms

As observed above, we have that the ω map monotonically preserves simplicity, but that the exp map does not (as will be made clearer once we have log defined). However, there are a subclass of numbers where exp is a monotonic map preserving simplicity, as the following theorem from [?] shows

Theorem 21. *If* $a, b \in \mathbb{J}$ *and* $a \trianglelefteq b$*, then* $\exp(a) \leqslant_s \exp(b)$*.*

In general, the above result is not sufficient for studying exp and $\leqslant_s$. The authors [?] remedied this by introducing the notion of **nested truncation** and a corresponding rank.

Definition. *A finite sum of surreal numbers* $y = x_1 + x_2 + \cdots + x_n$ *is in* ***standard form*** *if* $S(x_1) > S(x_2) > \cdots > S(x_n)$*.*

For $x \in \mathrm{No}^{\times}$*, set* $\mathtt{sgn}(x) = 1$ *if* $x > 0$ *and* $\mathtt{sgn}(x) = -1$ *otherwise.*

We then inductively define ranks $\underset{-n}{\blacktriangleleft}$ *on* $\mathrm{No}^{\times}$ *over* $n \in \omega$ *as follows:*

1. $x \underset{-0}{\blacktriangleleft} y$ *if* $x \trianglelefteq y$*;*
2. $x \underset{-n+1}{\blacktriangleleft} y$ *if there are* $a \underset{-n}{\blacktriangleleft} b$ *with* $a, b \in \mathbb{J}^*$*, and* $z, w \in \mathrm{No}$ *and* $r \in \mathbb{R}^{\times}$ *such that*

$$x = z + \mathtt{sgn}(r) \exp(a)$$

$$y = z + r \exp(b) + w$$

where both sums are in standard form.

*We say $x \underline{\blacktriangleleft} y$, or that x is a **nested truncation of y** if there is an n such that $x \underline{\blacktriangleleft}_n y$.*

$\underline{\blacktriangleleft}$ induces a foundation rank, which we define as follows:

Definition. *For all $x \in \mathrm{No}^\times$, the **nested truncation rank**, $NR(x)$ is defined by*

$$NR(x) := \sup\{NR(y) + 1 \mid y \underline{\blacktriangleleft} x$$

With $NR(0) = 0$

Remark. *Since all real numbers have no proper truncations, we find that $\mathbb{R}$ has nested truncation rank 0*

Theorem 22. *$\underline{\blacktriangleleft}$ partially orders $\mathrm{No}^\times$.*

Proof. It is immediate that $\underline{\blacktriangleleft}$ is reflexive since $x \trianglelefteq x$ for all x.

We prove antisymmetry as follows. Suppose for some n that $x \underline{\blacktriangleleft}_n y$ and $y \underline{\blacktriangleleft} x$. Immediately o.t. S(x) = o.t. S(y). Proceeding by induction on n, for n=0, we have $x = y$. For $n > 0$, write x and y in the standard forms as above with $a \underline{\blacktriangleleft}_{n-1} b$. By the observation on order types, we have that $w = 0$ and by the hypothesis that $y \underline{\blacktriangleleft} x$, we have that $r = \mathtt{sgn}(r)$, and $b \underline{\blacktriangleleft} a$. But then by our inductive hypothesis, we have $a = b$, and hence $x = y$.

We prove transitivity as follows: Supposing for $n, m \in \omega$ that $x \underline{\blacktriangleleft}_n y \underline{\blacktriangleleft}_m z$. If $n = 0$, then $x \trianglelefteq y$, from which it follows that $x \underline{\blacktriangleleft}_m z$ as a truncation. Similarly, if $m = 0$, we have $y \trianglelefteq z$, from which $x \underline{\blacktriangleleft}_n z$. If $m, n > 0$, write y and z in the following standard forms

$$y = u + \mathtt{sgn}(r)\exp(b)$$

$$z = u + r\exp(c) + w$$

with $b \blacktriangleleft_{m-1} c$ (and \$b,c∈ J^*). We are done if $x \underline{\blacktriangleleft}_n z$, as $z \trianglelefteq u$ implies $x \underline{\blacktriangleleft}_n u$.

Otherwise, $\neg(x \underline{\blacktriangleleft}_n z)$, so we must have $x = z + \mathtt{sgn}(r)\exp(a)$ with $a \underline{\blacktriangleleft}_{n-1} b$ and $a \in \mathbb{J}^*$. By our induction hypothesis, we have that $a \underline{\blacktriangleleft} c$, from which $x \underline{\blacktriangleleft} u$. $\square$

[?] establish the following facts on $\underline{\blacktriangleleft}$:

Fact 5.

1. *For all $x, y \in \mathrm{No}^\times$ and $z \in \mathrm{No}$, if $z + x$ and $z + y$ are in standard form, then $x \underline{\blacktriangleleft} y \iff x + z \underline{\blacktriangleleft} y + z$.*

2. *For all $x \in \mathrm{No}^\times$, and $\mathfrak{m} \in \mathfrak{M}$, if $x \underline{\blacktriangleleft} \mathfrak{m}$, then $x \in \mathfrak{M}$.*

3. *$\underline{\blacktriangleleft}$ is the smallest transitive relation such that:*

- *for all* $x, y \in \mathrm{No}^{\times}$, $x \trianglelefteq y \Rightarrow x \underline{\blacktriangleleft} y$;
- *for all* $a, b \in \mathbb{J}^{*}, a \underline{\blacktriangleleft} b$ *implies that* $\exp(a) \underline{\blacktriangleleft} \exp(b)$ *and* $-\exp(a) \underline{\blacktriangleleft} -\exp(b)$;
- *for all* $\mathfrak{m} \in \mathfrak{M}^{\neq 1}$ *and* $r \in \mathbb{R}^{\times}$, $\mathtt{sgn}(r)\mathfrak{m} \underline{\blacktriangleleft} r\mathfrak{m}$;
- $\forall x, y \in \mathrm{No}^{\times}$, *and* $z \in \mathrm{No}$, *if* $z + x, z + y$ *are both in standard form, then if* $x \underline{\blacktriangleleft} y$, *then* $z + x \underline{\blacktriangleleft} z + y$.

4. *For all* $x \in \mathrm{No}$, *the class* $\{y \in \mathrm{No} \mid x \trianglelefteq y\}$ *is convex.*
5. *For all* $x \in \mathrm{No}^{\times}$, *the class* $\{y \in \mathrm{No}^{\times} xy\}$ *is convex.*
6. $\mathbb{J}$ *is closed under* $\trianglelefteq$ *and* $\underline{\blacktriangleleft}$. *Namely, for all* $x \in \mathrm{No}^{\times}$ *and* $a \in \mathbb{J}^{*}$, *if* $x \underline{\blacktriangleleft} a$, *then* $a \in \mathbb{J}^{*}$.
7. $x \underline{\blacktriangleleft} y$ *implies that* $x \leqslant_s y$, *implying that* $\underline{\blacktriangleleft}$ *is well-founded, so that* $\underline{\blacktriangleleft}$ *has an associated ordinal rank which we'll call our* ***nested tree rank***.

We state without proof several facts regarding the nested truncation rank:

Fact 6. 1. *For all* $x \in \mathrm{No}$, $NR(x) = NR(-x)$.

2. *For all* $a \in \mathbb{J}$, $NR(a) = NR(\pm \exp(a))$.
3. *For all* $\mathfrak{m} \in \mathfrak{M}^{\neq 1}$, *and* $r \in \mathbb{R}^{\times}$, *if* $r \neq \pm 1$, *then* $NR(r\mathfrak{m}) = NR(\mathfrak{m}) + 1 > NR(\mathfrak{m})$.
4. *If* $x \neq 0$, *and if* $r\mathfrak{m}$ *is a term of* x, *then* $NR(r\mathfrak{m}) \leqslant NR(x)$ *and if* $NR(\mathfrak{m})$ *is not minimal in* $S(x)$, *then* $NR(r\mathfrak{m}) < NR(x)$.

1.6 log

1.6.1 Introducing the g function

While we had shown that for $a > 0$, $\exp(\omega^a)$ is of the form ω^{ω^b} without using inductionl, we actually are able to inductively define $g(a)$ using *orders of magnitude*, as follows:

Theorem 23. *Letting* $a = \sum_{\alpha} \omega^{a_i} r_i$, *we set* $c = a_0$, *i.e.* c *is the unique surreal number such that* $a \sim \omega^c$. *Then*

$$g(a) := \{c, g(a_L)\} \mid \{g(a_R)\}$$

Proof. From our earlier work, we have for positive surreal numbers x that

$$\exp(\omega^x) = \omega^{G(x)}$$

so we identify $G(a) = \omega^{g(a)}$ by the theorem where g was introduced.

Now, we define $G(0) = 0$, and then from the proof where g was introduced, by cofinality we find that

$$G(a) := \{rG(a_L) + na\} \mid \{sG(a_R)\} = \{0, rf(a_L) + na\} \mid \{sf(a_R)\}.$$

Then, by inductively substituting $G(a^o) = \omega^{g(a^o)}$, we obtain

$$G(a) = \{0, na, r\omega^{g(a_L)} + na\} \mid \{s\omega^{g(a_R)}\}.$$

Since na will be equicofinal with $n\omega^c$ and $r\omega^{g(a_L)} + na$ will be equicofinal with $r\omega^{g(a_L)} + n\omega^c$ and thus equicofinal with

$$n\omega^{\max(g(a_L),c)},$$

we find that

$$G(a) = \{0, n\omega^c, n\omega^{\max(g(a_L),c)}\} \mid \{s\omega^{g(a_R)}\}.$$

Thus, with $G(a) = \omega^{g(a)}$, by our definition of the ω map and cofinality we have

$$g(a) = \{c, \max(g(a_L), c)\} \mid \{g(a_R)\} = \{c, g(a_L)\} \mid \{g(a_R)\}$$

□

Example 2. *We quickly verify that* $\exp(\omega^\varepsilon) = \omega$ *using* g *as follows. By induction, assume that* $g(2^{-n}) = 2^{-n}$. *Then*

$$g(2^{-n-1}) = g(\{0\} \mid \{2^{-n}\}) = \{0\} \mid \{g(2^{-n})\} = \{0\} \mid \{2^{-n}\} = 2^{-n-1}$$

whence

$$g(\omega^{-1}) = g(\{0\} \mid \{2^{-n}\}) = \{-1\} \mid \{2^{-n}\} = 0.$$

Thus

$$\exp(\omega^\varepsilon) = \omega^{\omega^0} = \omega.$$

Remark. *So far, the definition of* g *does not seem to have warranted my earlier comments about the difficulties of tersely describing* g. *There are plenty of results to come which ought to justify my comments; suffice to say that* g *is not an identity function due to* $a \sim \omega^c$. *In fact,* $g(x)$ *may take on negative values, which is entirely consistent with* ω^{ω^x} *being positively infinite for all* $x \in \mathrm{No}$.

However, the main reason that it has taken awhile to introduce g *is that the study of the epsilon numbers in* No, *and particularly the behavior of these numbers under the exp map, is closely tied to the* g *function. Furthermore, many of the results pertaining to the* g *function require our still as of yet unstated and unproven sign representation lemma. Again, we shall defer studying both until we have some more results pertaining to our* exp *and* log *functions, and the questions raised in the past few papers.*

1.6.2 Exponential and Natural Log Revisited

The original reason we introduced g in the first place was to study how exp transforms normal forms. Towards that end we will prove the following generalized linearity property for exp:

Theorem 24. *If $a_i > 0$ for all $i \in \alpha$, then*

$$\exp(\sum_\alpha \omega^{a_i} r_i) = \omega^y$$

where

$$y = \sum_\alpha \omega^{g(a)_i} r_i$$

with $g(a)_i = g(a_i)$.

Proof. Since $\exp x$ and ω^x are both homomorphisms, this follows immediately for all finite sums and rational r_i. From here, we proceed in stages.

First, for monomials $\omega^a r = \{\omega^a r_L\} \mid \{\omega^a r_R\}$, where r^o are given as some dyadic representation, by induction and the density of the dyadic representations in $\mathbb{R}$, we have that

$$\exp(\omega^a r) = \{0, \exp(\omega^a r_L)_n[\omega^a r - \omega^a r_L]\} \mid \{\frac{\exp(\omega^a r_R)}{_n[\omega^a r_R - \omega^a r]}\}.$$

We then simplify the representatives by mutual cofinality to

$$\exp(\omega^a r) = \{0, \omega^{\omega^{g(a)} r_L + na}\} \mid \{\omega^{\omega^{g(a)} r_R - na}\}.$$

Hence, we have

$$\omega^{\omega^{g(a)}} > \omega^{na}$$

from which

$$\omega^{g(a)} > na$$

follows in general for all positive integers. Thus

$$\omega^{g(a)} > \frac{n}{r - r_L} a \equiv \omega^{g(a)} r - \omega^{g(a)} r_L > na,$$

whence

$$\omega^{g(a)} r_R - na > \omega^{g(a)} r > \omega^{g(a)} r_L + na.$$

Having satisfied the inbetweenness condition and since the lower terms have no maximum and the upper terms have no minimum, by cofinality we find that

$$\omega^{\omega^{g(a)} r} := \{0, \omega^{\omega^{g(a)} r_L}\} \mid \{\omega^{\omega^{g(a)} r_R}\}$$

.

We now proceed to induct on α for arbitrary sums.

The non-limit cases follow immediately by the additive properties of the exp and ω maps.

Supposing that α is a limit ordinal, then for arbitrary $\gamma \in \alpha$ and finite $s > 0$,

$$\sum_{\alpha} \omega^{a_i} r_i = \{\sum_{\gamma} \omega^{a_i} r_i - \omega^{a_\gamma} s\} \mid \{\sum_{\gamma} \omega^{a_i} r_i + \omega^{a_\gamma} s\},$$

whence

$$\exp(\sum_{\alpha} \omega^{a_i} r_i) = \{0, \exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{\alpha_\gamma} s)(\omega^{a_\gamma} \sigma)^n\} \mid \{\exp(\sum_{\gamma} \omega^{a_i} r_i + \omega^{a_\gamma})(\omega^{a_\gamma} \rho)^{-n}\},$$

where σ (and similarly ρ) is such that

$$\omega^{a_\gamma} \sigma = \omega^{\alpha_\gamma} s + \sum_{\alpha \setminus \gamma} \omega^{a_i} r_i$$

i.e. $|s - \sigma|, |s - \rho|$ will be infinitesimal.

Furthermore,

$$\sum_{\alpha} \omega^{g(a)_i} r_i = \{\sum_{\gamma} \omega^{r_{g(a)} -} {}_{g(a)} \omega^{g(a_\gamma)} s\} \mid \{\sum_{\gamma} \omega^{g(a)_i} r_i + \omega^{g(a_\gamma)} s\},$$

and since the lower terms have no maximum and the upper terms have no minimum, we find that

$$\omega^{\sum_{\alpha} \omega^{g(a)_i} r_i} = \{0, \omega^F\} \mid \{\omega^G\}$$

where F, G are the set of lower and upper terms respectively.

As is common in all of these proofs, we will use cofinality to show that the representation of $\exp(\sum_{\alpha} \omega^{a_i} r_i)$ will give $\omega^{\sum_{\alpha} \omega^{g(a)_i} r_i}$ after first verifying the betweenness condition.

The betweenness condition follows by mutual cofinality and several obvious substitutions such as $\omega^{g(a)} > na$ for all $n \in \mathbb{Z}$, and from s not being an infinitesimal. Specifically, a common lower term will be

$$\exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{a_\gamma} s) \omega^{n a_\gamma} = \omega^y$$

where $y = \sum_{\gamma} \omega^{g(a)_i} r_i - \omega^{g(a_\gamma)} s + n a_\gamma$ by the inductive hypothesis and the additivity of exp.

We then see the betweenness for lower terms is satsfied as

$$\omega^y < \sum_{\gamma} \omega^{g(a)_i} r_i - \omega^{g(a_\gamma)} \frac{s}{2} < \sum_{\gamma} \omega^{g(a)_i} r_i,$$

and a similar inequality holds for the upper terms, so that by the inductive hypothesis, a typical term of ω^F is of the form $\exp(\sum_{\gamma} \omega^{a_i} r_i - \omega^{\alpha_\gamma} s)$. Since $a > 0$ by hypothesis, we have that $\omega^{na} \geqslant 1$ and this completes the proof for representatives of ω^F. A similar argument is run for ω^G. $\square$

Remark. *As a consequence of this result, studying the behavior of* $\exp x$ *reduces to studying* g.

1.6.3 The Uniformity of the Natural Log

Having seen that the task of studying exp reduces to studying g, we naturally ought to ask is there something similar that we can use to study log?

The answer is yes, but before we define an h function that acts as the inverse of g, we need to check that we can obtain $\log(\omega^a)$ using representations of $a = \{a_L\} \mid \{a_R\} = F|G$.

The uniformity theorem is valid for the natural log function.

Proof. The following inequalities are derived from standard order of magnitude arguments and properties we have established about the ω map:

For lower elements $a_L < x < a$, we have $\log(\omega^x) + n \geqslant \log(\omega^{a_L}) + n$ and $\log(\omega^x) + \omega^{\frac{a-x}{n}} \leqslant \log(\omega^{a_L}) + \omega^{\frac{a-a_L}{n}}$.

For upper elements $a < x < a_R$,$\log(\omega^x) + n \leqslant \log(\omega^{a_R}) + n$ and $\log(\omega^x) - \omega^{\frac{x-b}{n}} \geqslant \log(\omega^{b_R}) - \omega^{\frac{a_R-a}{n}}$.

Once we have these inequalities have been established, as with all uniformity theorem proofs, the rest of the proof is handled by the use of the inverse cofinality theorem and an application of the cofinality theorems. ☐

1.6.4 TODO $\log : \mathbf{No}^+ \twoheadrightarrow \mathbf{No}$

Theorem 25. *For all* $a \in \text{No}$, $\ln(\omega^{\omega^a})$ *is a power of* ω.

Proof. By uniformity, we look at the representation $\omega^a = \{0, \omega^{a_L r}\} \mid \{\omega^{a_R s}\}$, to find the following after several simplifications, the additivity of log courtesy of what we (currently) know about exp (specifically that on the domain under investigation that log is an inverse of exp, and thus ln is additive in the familiar sense: $\log(\omega^a) + \log(\omega^b) = \log(\omega^{a+b})$), and general cofinality arguments:

$$\begin{aligned}
\log(\omega^{\omega^a}) &= \{\log(\omega(0)) + n, \log(\omega^{\omega^{a_L} r}), \log(\omega^{\omega^{a_R} s}) - \omega^{\frac{\omega^{a_R} s - \omega^a}{n}}\} \mid \{\log(\omega^{\omega^{a_R} s}) - n, \log(\omega^o) + \omega^{\omega^a - 0} n, \log(\omega^{\omega^a - \omega^{a_L} r} n)\} \\
&= \{n, r\log(\omega^{\omega^{a_L}}) + n, s\log(\omega^{\omega^{a_R}}) - \omega^{\frac{\omega^{a_R} s - \omega^a}{n}}\} \mid \{s\log(\omega^{\omega^{a_R}}) - n, \omega^{\omega^a}, r\log(\omega^{\omega^{a_L}}) + \omega^{\frac{\omega^a - \omega^{a_L} r}{n}}\} \\
&= \{n, r\log(\omega^{\omega^{a_L}}), s\log(\omega^{\omega^{a_R}}) - \omega^{\frac{\omega^{a_R} s - \omega^a}{n}}\} \mid \{s\log(\omega^{\omega^{a_R}}), \omega^{\omega^{a/n}}, r\log(\omega^{\omega^{a_L}}) + \omega^{\frac{\omega^a - \omega^{a_L} r}{n}}\} \\
&= \{n, r\log(\omega^{\omega^{a_L}})\} \mid \{s\log(\omega^{\omega^{a_R}}), s\omega^{\omega^{a/n}}\}
\end{aligned}$$

That is, by cofinality, the final representation of $\log(\omega^{\omega^a})$ exhibits a surreal number of the form ω^x. Moreover, we may define an 'inverse of g(x)' as follows:

$$\omega^{h(x)} = \ln(\omega^{\omega^x})$$

such that

$$h(a) := \{0, h(a_L)\} \mid \{h(a_R), \omega^{\frac{a}{n}}\}.$$

☐

Remark. *We have that for all* $a \in \mathrm{No}$*,* $h(a) > 0$*, and* this *is what shows that the range of the g function consists of all the surreal numbers, whence we may conclude that* $\exp x$ *induces a map from the class of the positive surreal numbers onto the class of all surreal numbers via the* $\ln$ *map.*

Furthermore, the uniformity theorem is valid for g and g, as for $x \leqslant y$*, then* $c_x \leqslant c_y$*, where* c_x *is such that* $x \sim \omega^{c_x}$ *and similarly for* c_y*.*

1.7 log-atomic numbers

The following is a rapid overview of the log-atomic numbers and their properties, see [?] for further details.

Definition. *Let x be a positive infinite surreal number. x is* ***log-atomic*** *if for all* $n \in \omega$*,* $\log_n x \in \mathfrak{M}^{>1}$*, i.e the* n^{th} *log iterate is an infinite monomial for all natural numbers n. Let* $\mathbb{L}$ *denote the class of log-atomic numbers. It follows that* $\mathbb{L} \subset \mathfrak{M}^{>1}$*.*

Berarducci-Mantova introduced a weaker order relation than the one tracking Archimedean class:

Definition. *For* $x, y \in \mathrm{No}$*, with* $x, y > \mathbb{N}$*,*

1. $x \preceq^{\mathbb{L}} y$ *if* $x \leqslant \exp_n(k \log_n(y))$ *for some* $n, k \in \mathbb{N}^+$*;*
2. $x \prec^{\mathbb{L}} y$ *if* $x \leqslant \exp_n(\frac{1}{k} \log_n(y))$ *for all* $n, k \in \mathbb{N}^+$*;*
3. $x \asymp^{\mathbb{L}} y$ *if* $\exp_n(\imath k \log_n(y)) \leqslant x \exp_n(k \log_n(y))$ *for some* $n, k \in \mathbb{N}^+$*.*

One can check that $\asymp$ *is an equivalence relation. We say that the equivalence class*

$$[x] = \{y \in \mathrm{No} \mid y > \mathbb{N} \wedge y \asymp^{\mathbb{L}} x\}$$

is the ***level*** *of x.*

The following facts can be found in [?]:

Fact 7.

1. $\asymp^{\mathbb{L}}$ *is an equivalence relation with* $x \asymp^{\mathbb{L}} y$ *if and only if there exists an* $n \in \mathbb{N}$ *such that* $\log_n(x) \sim \log_n(y)$*.*
2. *Each level of x is a union of positive parts of archimedean classe and* $\preceq^{\mathbb{L}}$ *induces a total order on the levels.*
3. *For all* $\mu, \lambda \in \mathbb{L}$*, if* $\mu < \lambda$*, then* $\mu \prec^{\mathbb{L}} \lambda$*.*
4. *If* $x, y > \mathbb{N}$*, and* $x \trianglelefteq y$*, then* $x \asymp^{\mathbb{L}} y$*.*
5. $\mathbb{L}$ *is a class of representatives for* $\asymp^{\mathbb{L}}$ *with each* $\lambda \in \mathbb{L}$ *the simplest number in its level (with respect to* $\leqslant_s$*.*
6. *For all* $x \in \mathrm{No}$*, NR(x)=0$ if and only if* $x \in \mathbb{R}$ *or* $x = \pm\lambda^{\pm 1}$ *for some* $\lambda \in \mathbb{L}$*.*

1.7.1 λ numbers

Recalling ℓ as the Krull valuation defined above (and that the surreal numbers form their own value group), we have the following consequence of some of the facts above:

Proposition 4. *For any $x \in \mathrm{No}$ such that $\ell(x) \neq 0$, there is some $n \in \omega$ such that $\ell_n(x) = \ell(\cdots(\ell(x)) \in \mathbb{L}$.*

We can parametrize the levels of $\mathbb{L}$ with the so called λ numbers, which have a genetic definition

Definition. *For every $x \in \mathrm{No}$ with canonical representatives x^L, x^R, define*

$$\lambda(x) := \{k, \exp_n(k \log_n(\lambda(x^L)))\} \mid \{\exp_n(\frac{1}{k} \log_n(\lambda(x^R)))\}$$

where n, k range over ω.

Question 1. *What are the cardinal characteristics of $\log(\alpha)$ for $\alpha \in \mathrm{On}$.*

Fact 8.

1. *$\lambda : \mathrm{No} \to \mathrm{No}$ is a well-defined monotonically increasing map such that $x < y \to \lambda(x) <^{\mathbb{L}} \lambda(y)$.*
2. *For every $x \in \mathrm{No}$ with $x > \mathbb{N}$, there is a unique $y \in \mathrm{No}$ such that $x \asymp^{\mathbb{L}} \lambda(y)$ and $\lambda(y) \leqslant_s x$, with $\lambda(y)$ the simplest representative of its level.*
3. *$\lambda(\mathrm{No}) = \mathbb{L}$*

1.7.2 TODO κ numbers pt 2o

Recalling that the κ numbers are intended to convey a notion of magnitude, [?] define the following relation:

Definition. *For any two $x, y \in \mathrm{No}$ such that $x, y > \mathbb{N}$:*

1. *$x \preceq^{\kappa} y$ if $x \leqslant \exp_n(y)$ for some $n \in \mathbb{N}$;*
2. *$x \prec^{\kappa} y$ if $x < \log_n(y)$ for all $n \in \mathbb{N}$;*
3. *$x \asymp^{\kappa} y$ if $\log_n(y) \leqslant x < \exp_n y$ for some $n \in \mathbb{N}$.*

Proposition 5. *$\asymp^{\kappa}$ is an equivalence relation.*

Proposition 6. *For all $x, y \in \mathrm{No}$, with $x, y > \mathbb{N}$, $x \asymp^{\mathbb{L}} y$ implies $x \asymp^{\kappa} y$.*

We then properly define the κ numbers with respect to a genetic function that identifies canonical representatives of each $\asymp^{\kappa}$ equivalence class:

Definition. *For all $x \in \mathrm{No}$,*

$$\kappa(x) := \{\exp_n(0), \exp_n(\kappa(x^L))\} \mid \{\log_n(\kappa(x^R))\}$$

where n ranges over $\mathbb{N}$.

Remark. *It is seen immediately that* $\kappa(0) = \omega(0)$ *and* $\kappa(1) = \varepsilon(0)$.

Fact 9.
1. $x \leqslant_s y$ *if and only if* $\kappa(x) \leqslant_s \kappa(y)$.
2. *For all* $x > \mathbb{N}$, *there exists* $\kappa(y) \leqslant_s x$ *such that* $\kappa(y) \asymp^{\kappa} x$, *so each* $\kappa(y)$ *is the simplest element in its respective equivalence class.*
3. $x < y$ *implies that* $\kappa(x) \prec^{\kappa} \kappa(y)$.
4. $\log_n(\kappa(x))$ *is always of the form* $\omega(\omega(y))$, *and therefore each* $\log_n(\kappa(x)) \in \mathfrak{M}$.
5. $\kappa(\mathrm{No}) \subset \mathbb{L}$.
6. *There are numbers in* $\mathbb{L}$ *which cannot be obtained from* $\kappa(\mathrm{No})$ *by finitely many applications of* log *and* exp

Following this last fact, with the goal of generating $\mathbb{L}$ from $\kappa(\mathrm{No})$, Berarducci and Mantova focus on the $\kappa(-\alpha)$ numbers for $\alpha \in \mathrm{On}$. Specifically

$$\kappa(-\alpha) = \mathbb{N} | \{\log_n(\kappa(-\beta) \mid n \in \mathbb{N}, \beta \in \alpha\}$$

will be the simplest positive number less than $\log_n(\kappa(-\beta))$ for all $n \in \mathbb{N}$ and $\beta \in \alpha$. From this, they find

Proposition 7. *The sequence* $\langle \kappa(-\alpha) \mid \alpha \in \mathrm{On} \rangle$ *is a decreasing and coinitial with the positive infinite numbers (i.e. every positive infinite number is greater than some* $\kappa(-\alpha)$, *and from this we find* $\mathbb{L}$ *is coinitial in the positive infinite numbers.*

1.8 TODO ∂_{BM}

Berarducci and Mantova provide a construction of a derivative ∂_{BM} such that gives $(\mathrm{No}, +, \cdot, \exp, \partial_{BM})$ is a Hardy type series derivation. More precisely, they equipped No with a derivation so that No is a Liouville closed H-field with ∂_{BM} surjective and sending infinitesimals to themselves.

We begin by defining surreal pre-derivatios $D_{\mathbb{L}}$ and surreal derivations D in such a way to make (No, D) an H-field, a generalized notion of a Hardy field. Afterwards, we define the Berarducci-Mantova derivative, explore some immediate facts and properties of the derivative, and . Afterwards, in an additional subsection, we provide an overview of some transcendence results, applications to the theory of transseries, and integration.

Definition. *A* ***(surreal) pre-derivation*** *is a map* $D_L : \mathbb{L} \to \mathbb{R}_{>0}\mathfrak{M}$ *such that*

1. $\log(D_{\mathbb{L}}(\lambda)) - \log(D_{\mathbb{L}}(\mu)) \prec \max\{\lambda, \mu\}$.
2. $D_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda) D_{\mathbb{L}}(\lambda)$ *for all* $\lambda, \mu \in \mathbb{L}$.

A ***surreal derivation*** *is a function* $D : \mathrm{No} \to \mathrm{No}$ *with the following properties:*

1. *(Leibniz rule): D(xy)=D(x)+D(y)*
2. *(strong additivity):* $D(\sum_{i\in\ I} x_i) = \sum_{i\in\ I} D(x_i)$\$ *for all summable sequences* $\langle x_i \mid i \in I\rangle$
3. *(compatibility):* $D(\exp(x)) = \exp(x)D(x)$
4. *(real constant field):* $\ker(D) = \mathbb{R}$
5. *(H-field): if* $x > N$*, then* $D(x) > 0$

The following facts are true for all surreal derivations D:

Fact 10. 1. *if* $1 \neq x \succ y$*, then* $D(x) \succ D(y)$*;*

2. *if* $1 \neq x \sim y$*, then* $D(x) \sim D(y)$*;*
3. *if* $1 \neq x \asymp y$*, then* $D(x) \asymp D(y)$\$
4. *For* $x, y \in \mathrm{No}$*, if* $x, y, x-y$ *are all positive infinite, then*

$$\log(D(x)) - \log(D(y)) \prec x - y \preceq \max\{x, y\}$$

Berarducci-Mantova define their derivation ∂_{BM} first by defining one on $\mathbb{L} \to \mathrm{No}_{>0}$, and then extending the definition to all of No by means of path-derivatives.

Definition. *For* $\lambda \in \mathbb{L}$*, with* α *ranging over the ordinals, let*

$$\partial_{\mathbb{L}} := \exp\left(-\sum_{\lambda \preceq^{\kappa} \kappa(-\alpha)} \sum_{i=1}^{\infty} \log_i(\kappa(-\alpha)) + \sum_{i=1}^{\infty} \log_i(\lambda)\right)$$

Since $\langle \log_i \lambda \rangle$ *is a strictly decreasing sequence of monomials, it is summable. Similarly,* $\langle \kappa(-\alpha)\rangle$ *is decreasing, so* $\langle \log_i(\kappa(-\alpha))$ *will also be summable. Furthermore, if* $\lambda = \kappa(-\alpha)$ *for some ordinal* α*, then the terms* $\log_i(\lambda)$ *cancel out, and we find that*

$$\partial_{\mathbb{L}}(\lambda) = \exp\left(\sum_{\beta<\alpha} \sum_{i=1}^{\infty} \log_i(\kappa(-\beta))\right)$$

with $\partial_{\mathbb{L}}(\omega(0)) = \partial_{\mathbb{L}}(\kappa(0)) = 1$*.*

We now define paths and path derivatives, before we define ∂_{BM} with respect to the pre-derivative $\partial_{\mathbb{L}}$.

Definition. *A* ***path*** *is an sequence* $P : \mathbb{N} \to \mathbb{R}^{\times}\mathfrak{M}$ *such that for every* $n \in \mathbb{N}$*,* $P(n+1)$ *is term of* $\ell(P(n))$*.*

$\mathcal{P}(x)$ *is the set of paths such that* $P(0)$ *is a term of* x*.*

Given a path P*, the* ***path derivative*** $\partial(P) \in \mathbb{R}\mathfrak{M}$ *is defined as follows:*

1. *if for some* $n \in \mathbb{N}$ *such that* $P(n) \in \mathbb{L}$*, set* $\partial(P) = \prod_{i<k} P(i) \cdot \partial_L(P(k))$*;*

2. *if for all* $n \in \mathbb{N}$, $P(n) \notin \mathbb{L}$, *set* $\partial(P) = 0$.

We define the ***Berarducci-Mantova*** *derivative* $\partial : \mathrm{No} \to \mathrm{No}$ *by*

$$\partial(x) := \sum_{P \in \mathcal{P}(x)} \partial(P)$$

Given $x \in \mathrm{No}\backslash\mathbb{R}$, *the* ***dominant path*** *of* x *is the path* $Q \in \mathcal{P}(x)$ *such that* $Q(0)$ *is the term of maximum non-zero* ℓ *value of* x *and* $Q(i+1)$ *is the leading term of* $\ell(Q(i))$ *for all* $i \in \mathbb{N}$.

We now state many facts about the pre-derivative, paths, and the Berarducci-Mantova derivative:

Fact 11. 1. *For all* $\lambda, \mu \in \mathbb{L}$, $\log(\partial_{\mathbb{L}}(\lambda)) - \log(\partial_{\mathbb{L}}(\mu)) \prec \max\{\lambda, \mu\}$

2. *For all* $\lambda \in \mathbb{L}$, $\partial_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda)\partial_{\mathbb{L}}(\lambda)$

3. *If* P *is a path, then* $1 \prec P(i+1) \preceq \log(|P(i)|) < P(i)$ *for all* $i > 0$.

4. *If* $t \preceq u$ *are both monomial terms, and* v *is a term of* $\ell(t)$ *but not* $\ell(u)$, *then* $v^n \prec \frac{u}{t}$ *for all* $n \in \mathbb{N}$.

5. *If* P, Q *are two paths such that* $\partial(P), \partial(Q) \neq 0$, *then if* $P(0) \preceq Q(0)$ *and* $P(1)^n \prec \dfrac{Q(0)}{P(0)}$ *for all* $n \in \mathbb{N}$, *then* $\partial(P) \prec \partial(Q)$.

6. *Extending Fact 5, if there exists an* n *such that for all* $m \leqslant n$, $P(m) \preceq Q(m)$, *and* $P(n+1)^k \prec \dfrac{Q(n)}{P(n)}$ *for all* $k \in \mathbb{N}$, *then* $\partial(P) \prec \partial(Q)$.

7. *If* P,Q *are two paths with non-zero path derivative and there exists an* $n \in \mathbb{N}$ *such that for all* $m \leqslant n$, $P(m) \preceq Q(m)$ *and* $P(n+1)$ *is not a term of* $\ell(\mathbb{Q}(n))$, *then* $\partial(P) \prec \partial(Q)$.

8. *Given* $P \in \mathcal{P}(x)$, $NR(P(0)) \leqslant NR(x)$, *and if* $NR(P(0)) = NR(x)$, *then the minimum* $\mathfrak{m}$ *of* $S(x)$ *is such that* $P(0) = r\mathfrak{m}$ *for some* $r \in \mathbb{R}^{\times}$.

9. *Similarly, for all* $n \in \mathbb{N}$, $NR(P(n+1)) \leqslant NR(P(n))$ *and if equality holds, then there is a minimum* $\mathfrak{m}$ *in* $S(\ell(P(n)))$ *such that* $P(n+1) = r\mathfrak{m}$ *for some* $r \in \mathbb{R}^{\times}$.

10. *For all* $x \in \mathrm{No}$, *there is at most one path* $P \in \mathcal{P}(x)$ *such that* $NR(P(n)) = NR(x)$ *for all* $n \in \mathbb{N}$.

11. *If* $x \in \mathrm{No}\backslash\mathbb{R}$ *with dominant path* Q, *then* $\partial(Q) \neq 0$ *and* $\partial(Q)$ *is the leading term of* $\partial(x)$.

12. $\ker \partial = \mathbb{R}$.

13. *If* $x > \mathbb{N}$, *then* $\partial(x) > 0$

14. ∂ is strongly linear, and therefore strongly additive.

15. For all $\gamma \in \mathbb{J}$, ∂(exp(γ))=exp(γ)∂(γ)$.

16. For all $x, y \in$ No, $\partial(xy) = x\partial(y) + y\partial(x)$.

17. For all $x \in$ No, $\partial(\exp(x)) = \exp(x)\partial(x)$.

Using the facts above, we summarize the proof of summability from [**?**]

Theorem 26. *For all $x \in$ No, the family $\langle \partial(P) \mid P \in \mathcal{P}(x) \rangle$ is summable.*

Proof. For any $x \in$ No, it suffices to show that there is no sequence of distinct paths $\langle P_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{P}(x)$ such that we have an infinite ascending chain

$$\partial P_0 \preceq \partial P_1 \preceq \partial P_2 \preceq \cdots,$$

since $\partial(P) \in \mathbb{R}\mathfrak{M}$ for all $\mathcal{P}(x)$.

Towards a contradiction, suppose that there exists such a sequence and let $\alpha = NR(x)$. Since the paths are distinct, there must be a minimum $m \in \mathbb{N}$ such that $P_i(m) \neq P_j(m)$ for some $i, j \in \mathbb{N}$. We proceed by double induction, first on α, and then on m.

Let $r \exp(\gamma)$ be the maximum ℓ value from $\{P_j(0) \mid j \in \mathbb{N}\}$.

By fact 11.8, if $NR(\gamma) = \alpha$, then $r \exp(\gamma)$ is also the term of minimum ℓ value, whence $P_j(0) = P_0(0)$ for all j. Thus $m > 0$.

If $NR(\gamma) < \alpha$, we extract a subsequence so that

$$r \exp(\gamma) = P_0(0) \succeq P_1(0) \succeq P_2(0) \succeq \cdots.$$

If $P_j(1)$ is not a term of $\gamma = \ell(P_0(0))$ for some $j \in \mathbb{N}$, but Fact 11.7, we find that $\partial(P_j) \prec \partial(P_0)$, which is a contradiction.

Therefore, $P_j(1)$ must be a term of γ for all $j \in \mathbb{N}$.

Now consider paths Q_j defined by $Q_j(n) = P_j(n+1)$, for all $n \in \mathbb{N}$. Let r be the minimum integer such that $Q_j(r) \neq Q_k(r)$ for some j, k.

In the case of $NR(\gamma) = \alpha$, we have that $r = m - 1$, and that for all $j \in \mathbb{N}$, we have $Q_j \in \mathcal{P}(x)$.

Thus, we find that $\partial(P_j) = P_j(0) \cdot \partial(Q_j)$, and that we have a descending sequence

$$P_0(0) \succeq P_1(0) \succeq P_2(0) \succeq \cdots,$$

from which we derive an ascending sequence

$$\partial Q_0 \preceq \partial Q_1 \preceq \partial Q_2 \preceq \cdots.$$

Now, we either have that (1) NR(γ)=$\alpha$$ and $r < m$; or we have (2) $NR(\gamma) < \alpha$, and both of these contradict the induction hypothesis that no suchh sequence exists in γ.

Thus $\langle \partial P \mid P \in \mathcal{P}(x) \rangle$ is summable. □

Theorem 27. *∂_{BM} extends $\partial_{\mathbb{L}}$.*

Proof. By facts 11.12 to 11.17, we find that ∂_{BM} is a surreal derivation. By restricting ∂_{BM} to $\mathbb{L}$, $\partial_{BM} \upharpoonright \mathbb{L}$ takes values in the subfield $\mathbb{R}\langle\langle\mathbb{R}\rangle\rangle$ of No. Since we compute ∂_{BM} as finite products of infinite sums, we see that $\partial(\mathbb{R}\langle\langle\mathbb{R}\rangle\rangle) \subset \mathbb{R}\langle\langle\mathbb{R}\rangle\rangle$, from which $\partial_{BM} \upharpoonright \mathbb{L}\langle\langle\mathbb{L}\rangle\rangle$ will induce an H-field structure on $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$. □

Corollary 2. *Let $d : \mathbb{L} \to \mathrm{No}_{>0}$ be a map such that:*

1. *for all $\lambda, \mu \in \mathbb{L}$, $\log(d(\lambda)) - \log(D(\mu)) \prec \max\{\lambda, \mu\}$;*
2. *for all $\lambda(\mathbb{L}), d(\exp(\lambda)) = \exp(\lambda)d(\lambda)$;*
3. *$d(\mathbb{L}) \subset \mathbb{R}^{\times}\mathfrak{M}$.*

Then d extends to a surreal derivation D on No.

1.8.1 TODO Transcendence, Transseries, and Integration

1. Transcendence Recall that if V is a $\mathbb{Q}$ vector space, and $W \subset V$, then $H \subset V$ is a $\mathbb{Q}$**linearly independent modulo W** if its projection to V/W is $\mathbb{Q}$ linearly independent.

   Using Ax's theorem, and a general result regarding all models of $\mathbb{R}_{\exp}$, we can show that the definable closure operation coincides with exponential-algebraic closure. From this, the following Schaunel type statements will hold modulo the exponential-algebraic closure of the empty-set.

   Theorem 28. *For any $R \models \mathbb{R}_{\exp}$, if $x_1, \ldots, x_n \in R$ are $\mathbb{Q}$ linearly independent modulo dcl$(()\varnothing)$, and k is the exponential transcendence degree of $x_1, \ldots, x_n$ over dcl$(()\varnothing)$, then*

   $$tr.deg_{\ker(D)}(x_1, \ldots, x_n, E(x_1), \ldots, E(x_n)) \geqslant n + k$$

   Proofs for the above theorem can be found in 1001[?, ?]. This result can be restated for differential fields as Ax's theorem [**?**]:

   Theorem 29. *Suppose that (K, D) is a differential field, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are such that $D(x_i) = \dfrac{D(y_i)}{y_i}$ for all $i \leqslant n$. Furthermore, suppose that all x_i are $\mathbb{Q}$ linearly independent modul $\ker(D)$. Then*

   $$tr.deg_{ker(D)}(x_1, \ldots, x_n, y_1, \ldots, y_n) \geqslant n + 1$$

   Taking $K =$ No and $D = \partial_{BM}$, and $y_i = \exp(x_i)$ leads to the following corollary

   Corollary 3. *If $x_1, \ldots, x_n \in$ No are $\mathbb{Q}$ linearly independent modulo $\mathbb{R}$, then*

   $$tr.deg_{\mathbb{R}}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \ gen + 1$$

2. Transseries One important result from [?] was to disprove a conjecture of [?], namely, that $\mathrm{No} = \mathbb{R}\langle\langle\mathbb{R}\rangle\rangle$. This result relates to the study of fields of transseries, and relies on the well-foundedness of the partial order relation $\underline{\blacktriangleleft}$. Whereas $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ is a field containing $\mathbb{R}(\mathbb{L})$, and closed under infinite sums, exponentiation, and logarithm, it is nonetheless a proper subfield of No, which maintains a transseries structure.

   Before elaborating further on this result, we recall Schmelling's notion of a transseries.

   Definition. *Let F be an ordered field, and $\exp : (F, +) \to (F^\times, \times)$ be a monotonic increasing group homomorphism such that $\exp(x) \geqslant 1 + x$ for all $x \in F$ and $Im(\exp) = F_{>0}$. Further, let Γ be an ordered group, and $B \subset F((\Gamma))$ an additive group containing $F((\Gamma_{\leqslant 0}))$, with a monotonic homomorphism $\exp : (B, +) \to (F((\Gamma))^\times, \times)$ which extends $\exp : F \to F^\times$ to B.*

   *We say that $(F((\Gamma)), \exp)$ is a **field of transseries** if it satisfies the following four axioms: T1. $Im(\exp) = F((\Gamma))_{>0}$; T2. $\Gamma \subseteq \exp(F((\Gamma_{>0})))$; T3. $\exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ for all $x \in F((\Gamma_{<0}))$; T4. for all sequences of monomials $\mathfrak{m}_i \in \Gamma$, with $i \in \mathbb{N}$, such that for $r_{i+1} \in F^\times$ and for $\gamma_{i+1}, \delta_{i+1} \in F((\Gamma_{>0}))$,*

   $$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}),$$

   with $\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}$ in standard form, then there is a $k \in \mathbb{N}$ suh that $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$ for $i \geqslant k$.

   Taking $F = \mathbb{R}$ and $\Gamma = \mathfrak{M} = \exp(\mathbb{J})$), and $B = \mathrm{No}$, we find that $\mathrm{No} = \mathbb{R}((\mathfrak{M}))$ equipped with Kruskal's exponential function will be a model of 1-3.

   [?] argued that the well-foundedness of $\underline{\blacktriangleleft}$ is equivalent to No satisfying axiom T4. While the analysis of $\underline{\blacktriangleleft}$ in [?] is solely with respect to the surreal numbers and notions of paths therein, paths and $\underline{\blacktriangleleft}$ can be extended to general transseries structures as follows:

   Definition. *For a path $P : \mathbb{N} \to F^\times((\Gamma_{>0}))$, write*

   $$P(i) := r_i \exp(\gamma_{i+1} + P(i+1) + \delta_{i+1})$$

   where $\gamma_{i+1}, \delta_{i+1} \in \Gamma_{>0}$ and $\gamma_{i+1} + P(i+1) + \delta_{i+1}$ are in standard form.

   We define for all $x \in F((\Gamma))$ the notion of path space $\mathcal{P}(x)$ as before.

   *A path P **satisfies T4** if there exists a $k \in \mathbb{N}$ such that $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$ for all $i \geqslant k$. Otherwise, P **refutes T4**.*

   *$x \in F((\Gamma))$ **satisfies T4** if for all paths in $\mathcal{P}(x)$ satisfy T4. Otherwise, x refutes T4.*

Proposition 8. *Let $x \in \text{No}$, and $P \in \mathcal{P}(x)$. If $NR(P(i)) = NR(x)$ for all $i \in \mathbb{N}$ then P satisfies T4.*

Theorem 30. *Axiom T4 holds in* No*, with $F = \mathbb{R}$ and $\Gamma = \mathfrak{M}$, whence* No *is a transseries in the sense of Schmelling.*

The crucial point yet to be mentioned to understand the result at the top of this subsubsection is that axiom T4 is a weaker form of an axiom ELT4 introduced in [?].

Definition. *Let $\mathbb{F} \subseteq \text{No}$. $\mathbb{F}$ is* ***truncation closed*** *if for every $f \in \mathbb{F}$, and $\mathfrak{m} \in \mathfrak{M}$, we have $f \upharpoonright \mathfrak{m} \in \mathbb{F}$.*

A truncation closed subfield $\mathbb{F}$ of No *closed under logarithm satisfies ELT4 if and only if for all sequences of monomials $\mathfrak{m}_i \in \mathfrak{M} \cap \mathbb{F}$, with $i \in \mathbb{N}$, such that*

$$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}$$

where $r_{i+1} \in \mathbb{R}^{\times}; \gamma_{i+1}, \delta_{i+1} \in \mathbb{J}$ and $\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}$ is in standard form, there is a $k \in \mathbb{N}$ such that $r_{i+1} = 1$ and $\gamma_{i+1} = \delta_{i+1} = 0$ for all $i \geqslant k$.

[?] remarks that ELT4 implies that the sequence $(\mathfrak{m}_i)$ eventually satisfies $\mathfrak{m}_i \in \mathbb{L}$, and that in terms of paths, a truncation closed subfield $\mathbb{F}$ of No closed under log satisfies ELT4 if and only if for every $x \in \mathbb{F}$, and every path $\mathcal{P}(x)$, there exists a k such that $P(k+1) \in \mathbb{L}$. As a consequence, they prove the following proposition:

Proposition 9. *$\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ is the largest truncation closed subfield of* No *closed under* log *and satisfying ELT4.*

After proving this result, [?] provide a proof to show that $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ is a proper subclass of No.

3. **TODO** Integration

2 TODO Forcing

The notion of forcing was originally developed by Paul Cohen to construct a model of ZFC in which the Continuum Hypothesis did not hold. His approach took a transitive model M of ZFC and adjoined a generic set G such that $M[G] \models \neg\text{CH}$.

Throughout, we let $\mathbb{P} = (P, <)$ denote a non-empty partially ordered set, and call $(P, <)$ a **forcing notion** whose elements are **forcing conditions**. Conditions p and q are **compatible** if there exists an $r \leqslant p, q$, and otherwise, they are incompatible, which will be denoted by $p \perp q$. A set $W \subset P$ is an **antichain** if its elements are pairwise incompatible. A set $D \subset P$ is dense if for every $p \in P$, there is some $q \in D$ such that $q \leqslant p$.

Given a forcing notion $\mathbb{P}$ over some **ground model** M, let $G \subset P$ denote a generic filter over P. We then describe $M[G]$ as a **generic extension** of M. Each element in $M[G]$ has a **name** in M, and associated to $\mathbb{P}$ is a **forcing language**, and a **forcing relation** $\Vdash$. Given a generic set G, every constant of the forcing language is then interpreted as a constant in the generic extension $M[G]$.

The following facts can be found in [**?**] as Corollary 14.12 and Theorem 14.10 respectively.

Fact 12. *For every partially ordered set P, there is a complete Boolean algebra* $\mathbb{B} = B(P)$ *and a mapping* $e : P \to B^+$*, where* $(B^+, <)$ *is a separative partial order (i.e. for all* $p, q \in B^+$*, if* $p \nleqslant q$*, then there exists an* $r \leqslant p$ *incompatible with* q*), such that:*

1. *if* $p \leqslant q$*, then* $e(p) \leqslant e(q)$*;*

2. p *and* q *are compatible if and only if* $e(p) \cdot e(q) \neq 0$*;*

3. $\{e(p) | p \in P\}$ *is dense in* B*.*

B will be unique up to isomorphism.

Fact 13. *Let* $\mathbb{P}$ *be a separative partially ordered set. Then there is a complete algebra B such that:*

1. $P \subset B^+$ *and* $<$ *agree with the partial ordering of B*

2. P *is dense in B.*

The algebra B is unique up to isomorphism.

These two facts raise an interesting question given that No is a proper class.

Question 2. *Is there a complete (class) algebra* $\mathbf{B}$ *such that* No $\subset B^+$ *and* $<_s^o$ *agrees with the parital ordering of* $\mathbf{B}$*, and* No *is dense in* $\mathbf{B}$*?*

Answering this question will require a move into second order logic that can properly handle classes, and so we will put this question aside for now, as we continue to review the rudimentary elements of forcing.

Definition. *We inductively define* ***names*** *as follows:*

Let $M \models \mathrm{ZFC}$ *be a transitive model, let* $\mathbb{P} \in M$ *be a forcing notion. Then a* ***P-name*** σ *in M contains elements of the form* $\langle \tau, p \rangle$ *where* τ *is a P-name and* $p \in \mathbb{P}$*.*

Given a P-name σ *in M, and a P-generic filter over M, let*

$$\sigma_G := \{\tau_G \mid \exists p \in G, \langle \tau, p \rangle \in \sigma$$

and

$$M[G] := \{\sigma_G \mid \sigma \in M^P$$

where M^P *is the set of P-names.*

The following theorems have detailed proofs found in [**?**].

Theorem 31. *Let M be a transitive model of* ZFC*, and let $\mathbb{P}$ be a forcing notion in M. If $G \subset P$ is a generic filter over P, then there exists a transitive model $M[G]$ such that:*

1. *$M[G] \models$ ZFC;*
2. *$M \subset M[G]$ and $G \in M[G]$;*
3. *$\mathrm{On}^{M[G]} = \mathrm{On}^{M}$;*
4. *if N is a transitive model of* ZF *such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.*

The forcing relation $\Vdash$ generalizes model-theoretic satisfaction $\models$ in the forcing language.

Theorem 32. *Let $\mathbb{P}$ be a forcing notion in the ground model of M, and let M^P be the class in M of all names. Then*

1. (a) *If $p \Vdash \varphi$ and $q \leqslant p$, then $q \Vdash \varphi$;*
   (b) *No p forces φ and $\neg\varphi$;*
   (c) *For all p there is a $q \leqslant p$ such that $q \| \varphi$ (q **decides** φ), i.e. $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*
2. (a) *$p \Vdash \neg\varphi \iff \neg\exists q \leqslant p q \Vdash \varphi$*
   (b) *$p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi$ and $p \Vdash \psi$.*
   (c) *$p \Vdash \forall x\varphi \iff p \Vdash \varphi(\dot{a})$ for every $\dot{a} \in M^P$.*
   (d) *$p \Vdash \varphi \vee \psi \iff \forall q \leqslant p \exists r \leqslant q (r \Vdash \varphi$ or $r \Vdash \psi)$*
   (e) *$p \Vdash \exists x\varphi \iff \forall q \leqslant p \exists r \leqslant q \exists \dot{a} \in M^P (r \Vdash \varphi(\dot{a}))$.*
3. *If $p \Vdash \exists x\varphi$, then for some $\dot{a} \in M^P$, $p \Vdash \varphi(\dot{a})$.*

Example 3 (Cohen Forcing)**.** *Let $\mathbb{P} = (2^{<\omega}, <)$ with the ordering $q < p$ if and only if $p \sqsubset q$. Let M be a ground model containing $\mathbb{P}$, and let G be Pgeneric filter over M.*

Further, set $f = \bigcup G$. Since G is a filter, f will be a function whose domain is ω. Furthermore, we can regard f as a characteristic function on some subset $A \subset \omega$. This can be seen as follows:

For every $n \in \omega$, let $D_n = \{p \in 2^{<\omega} \mid n \in \mathtt{Dom} p\}$. It is immediate that D_n is dense in P, and therefore it will meet G for every $n \in \omega$. Thus $\mathtt{Dom} f = \omega$.

We note that $f \notin M$, and as a characteristic function of A, $A \notin M$ as well. For every Boolean function $g \in M$, set $D_g = \{p \in 2^{<\omega} \mid p \not\subset g\}$. It is immediate that D_g is also dense, so D_g will meet G, and thus $f \neq g$ as well.

The sets $A \subset \omega$ obtained above are known as **Cohen generic reals**. This leads to Cohen's famous theorem

Theorem 33. *There is a generic extension $V[G]$ such that $2^{\aleph_0} > \aleph_1$.*

Proof. Let P be the set of finite Boolean-valued partial functions defined on a subset of $\omega_2 \times \omega$, such that $p < q$ if $q \sqsubset p$.

If G is a generic set of conditions, set $f = \bigcup G$. Since G is a filter, f is a function.

We check that $\texttt{Dom}f = \omega_2 \times \omega$. Since the sets $D_{\alpha,n} = \{p \in P \mid (\alpha, n) \in \texttt{Dom}p\}$ will be dense in P, G will meet each $D_{\alpha,n\}}$. Thus $(\alpha, n) \in \omega_2 \times \omega$ for each $(\alpha, n) \in \omega_2 \times \omega$.

Now define $f_\alpha : \omega_\rightarrow\{0, 1\}$ for each $\alpha \in \omega_2$ by:

$$f_\alpha(n) = f(\alpha, n)$$

For $\alpha \neq \beta$, $f_\alpha \neq f_\beta$ since $D = \{p \in P \mid n \in \omega(p(\alpha, n) \neq p(\beta, n))\}$ is dense and therefore $G \cap D \neq \varnothing$.

Thus in $V[G]$, there is a monic map $\omega_2 \to 2^\omega$.

The remaining proof that $|\omega_2^V| = \aleph_2^V[G]$ (and that this forcing notion preserves cardinals) can be found in [?] Chapter 14. □

Each f_α above is a characteristic function of a Cohen generic real, and so P will adjoin $\aleph_2$ Cohen generic reals to the ground model.

3 TODO Some ordinal analysis

4 TODO What Next

Given the sign sequence lemma, and the understanding that each surreal number can be understood as a predicate of a given ordinal in Cantor normal form, and that forcing notions add no new ordinals, the behavior of the surreal numbers under various forcing notions warrants investigation.

It is worth noting that immediately, the field operations inductively defined over the surreals do not correspond nicely to boolean operations on the subsets of ordinals.

UP NEXT

- The surreals in Godel's constructible universe L.
- Ordinal Analysis for algebraic structures of interest (RCF, DRing, analytic field, etc);
- Classifying maps $j : \text{No} \to \text{No}$ with respect to elementary $j : V \to V[G]$;
- Friedman's Inner Model Hypothesis (does every first order sentence φ holding in an inner model of a universe $V^* \supseteq V$ hold in some inner model of V) - inner models of a model of ZF(C)$ are transitive submodels containing all of the ordinals (and therefore each inner model contains a copy of the surreals). This is inspired in part by the Inner model reflection principle that shows whenever a first order formula $\varphi(a)$ holds in V, then it holds in some inner model $W \subsetneq V$.

- Iterated forcings using subfields of the surreal numbers as the underlying posets.
- Studying generalizations of descriptive set theory to infinitary logics using singular, weakly, measurable, and supercompact cardinals κ and the subrings $\mathrm{No}(\kappa)$.