WHAT IS EQUIVALENCE?

0.1. Preliminaries: Weak $\omega-$groupoids as an Algebraic Structure

Is this an idea? It would be bad form to fail to mention type theory when speaking about ω-groupoids, if only because this would be analogous to talking about manifolds while failing to mention their utility in describing configuration spaces. This is because the syntactical approach gives a relatively concrete meaning to what is to follow. The following definitions are courtesy of Guillaume Brunerie:

Definition. A context Γ is **contractible** if either Γ is a singleton $\Gamma = (x : \star)$ or Γ is obtained from some Γ' by duplicating a variable $(x : A) \in \Gamma'$ and gluing $\Gamma = (\Gamma', (y : A), (x : x \simeq_A y))$.

Given a context Γ, a **term** is either a variable x such that $(x : A) \in \Gamma$ for some type A or a **coherence cell** $\texttt{coh}_{\Delta.A}(u_1, \dots, u_n)$, where $\texttt{coh}_{\Delta.A}$ is such that the input $\Delta = (x_1 : B_1, \dots, x_n : B_n)$ is a contractible context, the output A is a type in Δ, and in Γ, the arguments u_i are terms satisfying $u_k : B_k[x_1 := u_1, \dots, x_{k-1} := u_{k-1}]$ such that $\texttt{coh}_{\Delta.A}(u_1, \dots, u_n) : A[x_1 := u_1, \dots, x_n := u_n]$.

Example. If $a : \star$ is in Γ, then $\texttt{coh}_{(x:\star).(x \simeq_\star x)}(a) : a \cong_\star a$.

One of the remarkable things about weak ω-groupoids is that they can be used to algebraically encode geometric shapes. We remind the reader of the following definitions:

Definition. A **globe category** $\mathbb{G}$ is a category whose objects are $n \in \mathbb{N}$, and whose morphisms are generated by source maps $\sigma_n : n \to n+1$, and target maps $\tau_n : n \to n+1$, which satisfy the following relations

- $\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$
- $\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$

for all $n \in \mathbb{N}$. We define a **globular object** G in a category $\mathcal{C}$ as a contravariant functor $G : \mathbb{G} \to \mathcal{C}$. In particular, a **globular set** is a presheaf on $\mathbb{G}$, and the category of globular sets, denoted by $\texttt{gSet}$, is the category of all presheaves on $\mathbb{G}$. Thus, any globular set G corresponds to a collection of sets $\{G_n\}_{n \in \mathbb{N}}$ equipped with the **n-source and n-target** functions $\{s_n, t_n : G_{n+1} \to G_n\}_{n \in \mathbb{N}}$ where by contravariance of presheaves, the following **globular identities** holds for all $n \in \mathbb{N}$

- $G(\sigma_{n+1} \circ \sigma_n) = s_n \circ s_{n+1} = s_n \circ t_{n+1} = G(\tau_{n+1} \circ \sigma_n)$
- $t_n \circ s_{n+1} = t_n \circ t_{n+1}$

Definition. Recall that a **groupoid** G is a small[1] category such that for every pair of objects $x, y \in G$, every morphism $f \in \texttt{Hom}_G(x, y)$ is an isomorphism. More precisely, we mean each f is invertible.

A **weak ω-groupoid** G is a globular set such that for every contractible context Δ and types A in Δ, there is an operation $\texttt{coh}_{\Delta.A} : (\eta \in [\![\Delta]\!]) \mapsto ob([\![A]\!]_\eta)$ where

- $[\![\star]\!]_\eta := G$
- $[\![u \simeq_A v]\!]_\eta := \texttt{Hom}_{[\![A]\!]_\eta}([\![u]\!]_\eta, [\![v]\!]_\eta)$
- $[\![x]\!]_\eta := \eta(x)$
- $[\![\texttt{coh}_{\Delta.A}(u_1, \dots, u_n)]\!]_\eta := \texttt{coh}_{\Delta.A}([\![u_1]\!]_\eta, \dots, [\![u_n]\!]_\eta)$
- $[\![(x_1 : B_1, \dots, x_n : B_n)]\!] := \{(a_1, \dots, a_n) \mid a_k \in ob([\![B_k]\!]_{a_1, \dots, a_{k-1}})\}$

Notation. $[\![\bullet]\!]$ are the **Strachey brackets**, denoting a **semantic evaluation function**.

Remark. This syntactical approach interprets types as iterated hom globular sets of G and terms are cells of G.

Remark. The term *weak* refers to the notion that these invertible morphisms are restricted only to be invertible up to all higher equivalences. What we mean by equivalence will be answered in a moment.

0.2. What Is Equivalence? There are many instances in mathematics where we find ourselves asking the question, " Are two 'things' the 'same'?"

What is meant by 'things' and 'sameness' is really a matter of the working context of the question. For instance, it makes sense when working with sets to ask if two elements are the same, or if two sets are the same. This is tantamount to asking if $x = y$ or $X = Y$. Such a question is sensible because sets have a predicate structure amenable to this question. Sameness is a matter of membership; elements are either the same or they are different. But suppose we're asking questions about objects in a specific category, such as the category of groups $\texttt{Grp}$.

In this case, two objects X, Y can be the *same* object (as sets, $X = Y$), or they could be *isomorphic* (as there is some pair of maps $f \in \texttt{Hom}_{Grp}(X, Y), g \in \texttt{Hom}_{Grp}(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$). This distinction shouldn't be entirely unfamiliar since the contemporary conception of symmetry is built upon the idea that an object could be the same as itself but in different ways (we track this information by the group of automorphisms for a given object). However, we are only looking at objects. What about morphisms?

At the level of a category, morphisms are either the 'same', or they're different; there is no notion of isomorphic morphism when working within a category. However, when working within a 2-category, we introduce 2-morphisms, maps between morphisms

[1]A category is small if both the class of objects and each class of morphisms is a proper set.

satisfying certain composition rules, and it is these composition rules that allow us to identify when morphisms are the "same".[2] However, neither of these examples captures why it might be useful to have a coarser notion of equivalence than equality. Consider the category $\mathtt{Top}$, whose objects are topological spaces and whose morphisms are continuous maps. To my (rather limited) knowledge, topological spaces are not classified up to homeomorphism, but rather, a courser equivalence relation: **homotopy**. As a reminder, given two $X, Y \in ob\mathtt{Top}$ with $f, g \in \mathtt{Hom}_{\mathtt{Top}}(X, Y)$, we say f and g are homotopic, denoted by $f \sim g$, if there is a map $H \in \mathtt{Hom}(I \times X, Y)$ with $H(0, \cdot) = f$ and $H(1, \cdot) = g$, and where $I = [0, 1]$. We can then say two spaces are the *same* (more precisely, they are **homotopy equivalent**) if there are $f \in \mathtt{Hom}_{\mathtt{Top}}(X, Y)$ and $g \in \mathtt{Hom}_{\mathtt{Top}}$ which are inverses up to homotopy. By considering homotopies between our continuous functions, we've made $\mathtt{Top}$ into a 2-category, where the objects are still topological spaces, the 1-morphisms are still continuous maps, and the 2-morphisms are homotopies between these maps. Hopefully, this notion of equivalence is illuminating, as well as an ample illustration of just how we can have *levels* to our notion of equivalence, as we can proceed to take homotopies of our homotopies, and so on.

Using this iterated approach, we can regard $\mathtt{Top}$ as the prototypical ω-category. The fundamental reason for this seems innocuous enough; it is because I can be regarded as a reversible arrow object.[3] One of Grothendieck's many insights was that homotopy theory is a branch of higher category theory in the sense that for any space X, we can find a weak ω category $\Pi(X)$, whose objects are the points $x \in X$, whose 1-morphisms are paths, whose 2-morphisms are paths of paths, and so on. Because the unit interval is directed, every $k-$morphism in $\Pi(X)$ is an equivalence (that is, we can simply reverse the direction of the path). In this way, we can remark that $\Pi(X)$ is truly a weak ω-groupoid as all morphisms are invertible, and because composition is associative.

So to answer the motivating question, from the perspective of higher category theory, in a 0-category (the level of sets), an equivalence is merely equality. In a 1-category, an equivalence between objects is an isomorphism. In a two category, an equivalence between equivalent 1-morphisms is the existence of 2-cell isomorphisms, and so on.

0.3. Kan Complexes Now, we have previously seen the **globular presentation** of ω-groupoids. Another powerful example is the simplicial presentation where an $\omega-$groupoid is identified with **fibrant objects** in the simplicial category Δ.[4] This construction is typically called a **Kan complex**. In order to arrive at such a construction we will need the following definitions:

Definition. The **simplicial category** Δ is the category whose objects are non-empty, finite ordinal numbers and the order preserving maps between them.[5] In particular this means that the objects are sets $[n] = \{0, 1, \ldots, n\}$ and the hom-sets $Hom_{\Delta}([m], [n])$ are the non-strictly order preserving maps from $[m], [n]$.

There are two important subcategories of Δ, the category of injective order-preserving maps and the category of surjective order-preserving maps, denoted by Δ_+ and Δ_- respectively. We can generate Δ by the morphisms $d^i \in \mathtt{Hom}_{\Delta_+}([n-1], [n])$ and $s^i \in \mathtt{Hom}_{\Delta_+}([n], [n-1])$ where d^i 'skips' i and s^i identifies $i, i+1$.

Finally, a simplicial object in a category $\mathcal{C}$ is a presheaf on Δ to $\mathcal{C}$.

Example. We define a **simplicial set** $S_\bullet$ as a set valued presheaf on Δ. We can specify $S_\bullet$ by first specifying the corresponding sets S_n for all $n \in \mathbb{N}$, and then for each $f \in \mathtt{Hom}_{\Delta}([m], [n])$, specifying $S_\bullet f$ such that the compatability conditions on a functor are satisfied. Given a simplicial set $S_\bullet$, the set of **n-simplices** is denoted by $S_n := S_\bullet[n]$, the **face maps** are denoted by $d_i := S_\bullet(d^i) : S_n \to S_{n-1}$ for $n \geq 1$ and $1 \leq i \leq n$, and the **degeneracy maps** are denoted by $s_i := S_\bullet(s^i) : S_{n-1} \to S_n$. Given that a simplicial set is a pre-sheaf $S : \Delta^{op} \to \mathtt{Set}$, we can define the category of simplicial sets $\mathtt{sSet}$, whose objects are these presheafs, and natural transformations between them are morphisms. In particular, this means that $f : K \to S$ is a morphism of simplicial sets if $f_n : K_n \to S_n$ is a collection of maps which commute with the face and degeneracy maps.

Remark. There are a set of algebraic identities that characterize the relationship between d^i, s^i and d_i, s_ithat are respectively known as the *cosimplicial* and *simplicial* identities.

	Simplicial	cosimplicial
Category	$\mathtt{sSet}$	Δ
$i < j$	$d_i d_j = d_{j-1} d_i$ $d_i s_j = s_{j-1} d_i$	$d^j d^i = d^i d^{j-1}$ $s^j d^i = d^i s^{j-1}$
$i = j, j+1$	$d_i s_j = \mathrm{id}$	$s^j d^i = \mathrm{id}$
$i > j$ $i > j+1$	$s^j s^i = s^{i-1} s^j$ $d_i s_j = s_j d_{i-1}$	$s^j s^i = s^{i-1} s^j$ $s^j d^i = d^{i-1} s^j$

Example. Perhaps the most understandable simplicial set is the terminal simplicial set T which maps all $[n] \mapsto \{*\} = 1$, the terminal object in $\mathtt{Set}$. In this case, the identities hold vacuously, as they are all collapsed to the unique map $! : 1 \to 1$.

Example. We define the **standard n-simplex** by the contravariant functor $\Delta[n] := \mathtt{Hom}_{\Delta}(-, [n]) : \Delta \to \mathtt{Set}$, by $[i] \mapsto \mathtt{Hom}_{\Delta}([i], [n])$.[6] In particular, $\Delta[n]$ has $\binom{n}{k}$ non-degenrate $k-$simplices corresponding to the injective order-preserving maps $[i] \to [n]$, and one non-degenerate $n-$simplex, i_n. Given the standard n-simplices $\Delta[n]$, we can define the **internal Hom functor** $\Delta(-,-) : \Delta^{op} \times \Delta \to \mathtt{sSet}$. Furthermore, there is a natural isomorphism $\mathtt{Hom}_{\mathtt{sSet}}(\Delta[n], K) \cong K_n$ given by $f \mapsto f(i_n)$.

[2] 2-categories are not an entirely an unfamiliar notion. If one regards each group as a category of one object, $*$ equipped with a hom-set $G := \mathtt{Hom}(*, *)$ whose elements are invertible, then we could regard a 2-category of groups as a category whose objects are these various $\mathtt{Hom}(*, *)$, whose 1-morphisms are covariant functors between these hom-sets associated with normal group homomorphisms from $G \to H$, and whose 2-morphisms are simply *natural transformations*.

[3] Elsewhere in the literature, the key insight here is that I is a 1-dimensional **opetope**, which are the basic 'shapes' given of $k-$morphisms, and consequently, the shapes of the given coherence laws for composition.

[4] In a category $\mathcal{C}$, an object X is **fibrant** if the map $X \to 1$ is a fibration.

[5] i.e. $f \in \mathtt{Hom}_{\Delta}([n], [m]) \iff (\forall x, y \in [n], x \leq y \Rightarrow f(x) \leq f(y))$

[6] Similarly, for $g \in \mathtt{Hom}_{\Delta}([j], [k])$, $\Delta[n](g) : \mathtt{Hom}_{\Delta}([k], [n]) \to \mathtt{Hom}_{\Delta}([j], [k])$ is defined by $f \mapsto f \circ g$ for all $f \in \mathtt{Hom}_{\Delta}([k], [n])$

Example. To illustrate a rather silly morphism in $\mathtt{sSet}$, consider the trivial map from $\Delta[m]$ to T. Conversely, consider the family of maps from T to $\Delta[m]$. An equally vacuous morphism sends T to $\Delta[0]$. What about $f : T \to \Delta[1]$? Well, for each $n \in \mathbb{N}$, $f_n : T_n \to \mathtt{Hom}_\Delta([n],[1])$, which can be interpreted as specifying an ordered partition of $[n]$, as maps from $T_n = 1 = \{*\}$ are *elements*. In particular, this means that our choice of f consists of choosing a sequence of f_n, which are elements of $\mathtt{Hom}_\Delta([n],[1])$, such that

$$\begin{tikzcd} & * \arrow[dl, "f_n"'] \arrow[dr, "f_{n+1}"] & \\ \mathtt{Hom}_\Delta([n],[1]) \arrow[rr, "\Delta[1](s^i)"'] & & \mathtt{Hom}_\Delta([n+1],[1]) \end{tikzcd}$$

commutes for all n,i and similarly for our face maps. This requirement entails that $f_{n+1} = f_n \circ s^i$ and $f_n = f_{n+1} \circ d^i$

Definition. Given the standard simplex $\Delta[n]$, there are subsimplicial sets $\Lambda^i[n]$ called the (n,i) **horns** generated by taking the union of all faces of $\Delta[n]$ except for the i^{th} face.[7] Hinting at the geometry to come, $\Lambda^i[n]$ is obtained from $\Delta[n]$ by omitting the interior of $\Delta[n]$ and the interior of the $n-1$ dimensional face opposite to i. Whenever $i = 0$ or $i = n$, $\Lambda^i[n]$ is called an **outer horn**. Otherwise, it is an **inner horn**. Given a simplicial set S, a horn $\Lambda^i[n]$ of $\Delta[n]$ has a **filler** if given $\Lambda^i[n] \to S$, there is an extension e along the inclusion $\Lambda^i[n] \hookrightarrow \Delta[n]$ such that

$$\begin{tikzcd} \Lambda^i[k] \arrow[r] \arrow[d, hook] & S \\ \Delta[n] \arrow[ur, dotted, "e"'] & \end{tikzcd}$$

commute. This horn filling condition can be regarded as guaranteeing that given a horn $\Lambda^i[n]$, and the collection of all $i-1$ composable $i-$morphisms, there is an $i-$morphism (their composite), and an $n+1$ morphism connecting the original $n-1$ n-cells and their composite. If S is a simplicial set such that for every horn $\Lambda^i[n]$ there exists an extension such that the diagram commutes, then S satisfies the **Kan condition**. Given a simplicial set S, if S satisfies the Kan condition, then S is a **Kan complex**.

A **Kan fibration** is a morphism $\pi : S \to K$ of simplicial sets such that for any $n \geq 1$ and $0 \leq k \leq n$, π has the **right lifting property** for all horn inclusions, i.e. for every commuting square, there exists a lift from $\Delta[n] \to S$:

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & S \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & K \end{tikzcd}$$

0.4. Geometric Realization

We first recall the following definition:

Definition. From a categorical point of view, notions of geometry are inextricably linked to notions of topology (or rather, of topoi, which generalize topologies as being categories of sheaves on sites).[8]In particular, let us define a covariant functor $|\cdot| : \Delta \to \mathtt{Top}$ by sending $[n]$ to the standard topological n-simplex

$$\Delta_n = \{\mathbf{x} \in [0,1]^{n+1} \mid \mathbf{1} \cdot \mathbf{x} = 1\}$$

and by sending $f : [m] \to [n]$ to a continuous map $|f| = \varphi : \Delta_m \to \Delta_n$ defined by

$$\mathbf{x} = (x_0, \ldots, x_m) \mapsto \mathbf{y} = (y_j); \qquad y_j := \sum_{f(i)=j} x_i$$

Since $\mathtt{Top}$ is a co-complete category[9], there is an unique, induced colimit preserving functor $G_\bullet : \mathtt{sSet} \to \mathtt{Top}$ sending each standard $n-$simplex $\Delta[n]$ to Δ_n, with $f \mapsto \varphi$ as above. $|\cdot|$ is called the **geometric realization** functor, in part because it takes a *cellular shape* $[n]$ and *realizes* it as its corresponding standard *topological shape*, the standard $n-$simplex. In the literature, the geometric realization of a simplicial set $S_\bullet$ is denoted by $|S|$.

Example. While giving examples of useful functor, for $X \in ob\mathtt{Top}$, and $n \in \mathbb{N}$, the **singular n-simplex** in X is a continuous map $\sigma : \Delta_n \to X$, and we write

$$(\mathrm{Sing}X)_n := \mathtt{Hom}_{\mathtt{Top}}(\Delta_n, X)$$

for the set of singular $n-$simplices of X. We can then define a functor $\mathrm{Sing} : \mathtt{Top} \to \mathtt{sSet}$. It can be verified elsewhere that Sing is the right adjoint of $|\cdot|$; in particular, for each topological space X, and each $[n] \in \Delta$,

$$\mathrm{Sing}(X)([n]) = \mathtt{Hom}_{\mathtt{Top}}(|\Delta[n]|, X)$$

The punch line of all of this is to arrive at the following definition:

Definition. For $X, Y \in \mathtt{Top}$ or $\mathtt{sSet}$, $f : X \to Y$ is a **weak equivalence** if f induces the following isomorphisms:

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[7]In particular, this means that $\Lambda^i[n]$ is the simplicial subset of $\Lambda[i]$ generated by $\{d^0, \ldots, d^{i-1}, d^{i+1}, \ldots, d^n\}$

[8]Recall that a **site** for a category $\mathcal{C}$ is the pair $(\mathcal{C}, \mathcal{J})$, where $\mathcal{J}$ is the Grothendieck topology on $\mathcal{C}$, which is a collection of sieves $\mathcal{J}(X)$ for all objects $X \in ob\mathcal{C}$ subject to certain axioms

[9]We say a category is **co-complete**, or has **all small colimits**, if for every small category $\mathcal{S}$ and functor $F : \mathcal{S} \to \mathcal{C}$ a co-limit of F exists in $\mathcal{C}$. As a dual to the notion of a categorical limit, in abelian categories, co-limits generalize direct sums, and are sometimes referred to as **direct limits**. We further recall that where colimits of a diagram exist, they are the **co-classifying space** for morphisms *out* of the diagram.

- $\prod_0(f) : \Pi_0(X) \cong \prod_0(Y)$
- for all $x \in X$, and all $n \in \mathbb{Z}^+$, f induces an isomorphism on the homotopy groups $\pi_n(f;x) : \pi_n(X,x) \cong \pi_n(Y, f(x))$.