Motivations For Homotopy Type Theory

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Today

Key Ideas

- Categories
- Functors (and Fibrations in paticular)
- Natural Transformations
- Yoneda Lemma

A Motivating Object: Posets

- Given a set D, a partial order $\leq \subseteq D \times D$ satisfies:
  1. reflexivity
  2. antisymmetry
  3. transitivity
- We say $(D, \leq)$ above is a poset
- If $D \equiv (D, \leq)$ is a poset, we say a non-empty $X \subseteq_d D$ is a directed set if for all $x, y \in D$ there exists $z \in X$ such that $x \leq z \wedge y \leq z$
- $(D, \leq)$ is a complete partial order (CPO) if
  1. there is a least (or bottom) element $\bot \in D$ such that $\forall x \in D, \bot \leq x$
  2. For every $X \subseteq_d D$, a *supremum$ $\vee X \in D$ exists
- $(D, \leq)$ is a complete lattice if for every $X \subseteq D$ the supremum $\vee X$ exists in D.

Examples of complete lattices

Null

$(\bot, \leq)$ is vacuously a complete lattice

Bool

$(\{\bot, \top\}, \leq)$ such that $\bot \leq \top$ is a complete lattice

The class of ordinal numbers is a complete lattice

Powersets

For any set X, the power set of $X, \mathtt{P}(P)(X)$ is a complete lattice under inclusion

Scott Topology

- Given some CPO D, let $\Theta(D)$ be a topology on D such that $U \in \Theta(D)$
  1. if $x \in U$ and $x \leq y$, then $y \in U$
  2. if $X \subseteq_d$ and $\vee X \in U$, then $X \wedge U \neq \emptyset$
- Such a topology, $\Theta(D)$ is the Scott topology
- For any $x \in D$, set $U_x := \{z \in D \mid z \not\leq x\}$ is an open set.
- In general, D is T_0 but not T_1 by considering $x \neq y$ and $x \not\leq y$ and comparing U_x, U_y.

Prop: $f : D_1 \to D_2$ is continuous iff $f(\vee X) = \vee f(X)$ for all $X \subseteq_d D_1$

- We'll show that f is monotone first
- If f is continuous, towards a contradiction suppose that $x \leq_1 y$ and $f(x) \not\leq_2 f(y)$. Then $f(x) \in U_{f(y)}$ and therefore $x \in f^{-1}(U_{f(y)})$. Since $f^{-1}(U_{f(y)})$ is open, by the definition of the Scott topology, $y \in U_{f(y)}$, which is a contradiction. Hence f is monotone.
- The forward direction is proved by contradiction from the assumption that $f(\vee X) \not\vee f(X)$.
- In the reverse direction, given $x \leq y \Rightarrow f(x) \leq f(y)$, it follows $y = x \vee y$ and thus $f(y) = f(x) \vee f(y)$ so $f(x) \leq f(y)$, thus if $U \in \Theta(D_1)$, then $f^{-1}(U) \in \Theta(D_2)$

Categorical Notions: Categories

- A category C consists of two classes:
  1. A class of objects denoted $|C|$
  2. A class of morphisms denoted C
  3. A pair of maps $\texttt{Dom}, \texttt{Cod} : C \to |C|$ such that for every $f \in C$, $f \in C(\texttt{Dom}(f), \texttt{Cod}(f))$, the class of morphisms from $X = \texttt{Dom}f$ to $Y = \texttt{Cod}f$
  4. A composition operation typically denoted by $\circ$ (and diagrammatically denoted by ;) such that morphism classes denoted $C(X, Y)$ with $X, Y \in |C|$ satisfying the following axioms:
     1. every object $X \in |C|$, there is $1_X \in C(X, X)$.
     2. for every $f \in C(X, Y), g \in C(Y, Z)$ and $h \in C(Z, W)$,

$$1_Y \circ f = f = f \circ 1_X, \qquad h \circ (g \circ f) = (h \circ g) \circ f$$

  5.

- If $C(X, Y)$ is a set for all objects $X, Y \in |C|$, then C is locally small

Example 0

The empty (or initial) category has no objects and no morphisms, and so it vacuosly satisfies the categorical axioms. Moreover $0 = |0| := \{\}$ as ordinary notions of *sets*.

Example 1

The unit (or final) category consists of a single object $*$ and has only the identity map 1_*.

Example $\bullet_1 \rightrightarrows \bullet_1$

This category consists of two unlabelled objects and two morphisms s, t from one bullet to the other in addition to the unlabelled identity maps

Example FinSets

This category consists of the class of finite abstract sets with n elements denoted by $[n]$ and the class of all functions $f : [n] \to [m]$ for each $n, m \in \mathbb{N}$.

Example Sets

$|\text{Sets}|$ consists of the class of (small) sets in your favorite set theory, and the class of morphisms Sets consists of the sets $\text{Sets}(X, Y)$ which consist of set functions $f : X \to Y$, for every pair of sets $X, Y \in |\text{Sets}|$.

Example $\text{Sets}^{\to}$

$|\text{Sets}^{\to}|$ consists of all $f : X \to Y$, for every $X, Y \in |\text{Sets}|$; given $f : X \to Y$ and $g : Z \to W$, a morphism in $\text{Sets}^{\to}$ are pairs of set functions (h_g, h_f) such that $h_f \circ f = g \circ h_g$

Categories: More Examples

Posets

- Any poset $(P, \leq)$ may be regarded as a category, with $\leq$ being

the only morphism in each morphism class $P(x, y)$.

- We can form the category of posets, Pos whose objects are posets, and whose morphisms are order preserving maps

CPO

The category of complete partial orders is a *subcategory* of Pos whose objects are complete partial orders and whose morphisms are the order preserving maps.

$\Theta(X)$

Let X be a topological space with an *arbitrary* topology $\Theta(X)$. The open sets are the objects in this category, and morphisms are given by inclusion $\subseteq$.

- A graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, \sigma, \tau)$, with $\sigma, \tau : \mathcal{E} \to \mathcal{V}$.
- Categorically, $\mathcal{E} \equiv \mathsf{C}$ and $\mathcal{V} \equiv |\mathsf{C}|$
- Each category may be naturally viewed as a graph. We may also form a category Grph whose objects are graphs, and whose arrows are graph morphisms.

Categories: Pred

Not only are sets categories, but we can treat *predicates* on sets as a category as follows:

1. objects are pairs (I, X) such that $X \subseteq I$. We say that "X is a predicate of Y" and write $X(i)$ for $i \in X$. This choice of notation is intended to emphasize that $i \in I$ may be chosen as a *free variable*
2. morphisms $(I, X) \to (J, Y)$ are functions $u \in \text{Sets}(I, J)$ such that for all $i \in I$, $X(i)$ implies $Y(u(i))$

Just as we may turn predicates on sets into a category, we may also turn relations on sets into a category. We present the category of binary relations Rel as follows:

1. objects are pairs (I, R) where $I \in |\text{Sets}|$ and $RI \times I$
2. morphisms $(I, R) \to (J, S)$ are set functions $u \in \text{Sets}(I, J)$ such that for all $i, j \in I$, $R(i, j)$ implies that $S(u(i), u(j))$

- Objects are groups G, and morphisms are group homomorphisms f, e.g. $f(g_{1g}2) = f(g_1)f(g_2)$
- The category of abelian groups Ab is defined likewise.
- Given an abelian group G, $\mathrm{End}G := \mathrm{Ab}(G, G)$ can be used to define a ring, with $0 : g \mapsto 0$ and $1 : g \mapsto g$, addition given naturally and multiplication given by composition $\circ$ i.e. for all $x, y, z \in \mathrm{End}(G)$ and $g, h \in G$
  1. $(x + y)(g) = x(g) + y(g)$
  2. $x(g + h) = x(g) + x(h)$
  3. $x(y(g)) = (x \circ y)(g)$
  4. $((x \circ y) \circ z)(g) = (x \circ (y \circ z))(g) = x((y \circ z)(g)) = x(y(z(g))) = (x \circ y)(z(g)) = ((x \circ y) \circ z)(g)$

Categories: CRing

- Objects are commutative rings R, and morphisms are ring homomorphisms f, e.g. $f(ax + by) = f = (ax) + f(by) = f(a)f(x) + f(b)f(y)$
- Any $R \in \text{CRing}$ can be used to define a category R-mod.
- Recall that an R-module M is an abelian group M and an R-action defined by $\rho \in \text{CRing}(R, \text{End}(M))$ such that for any $r \in R$ and $m \in M$, $rm := \rho(r)(m)$.

- Objects are (left) R-modules M and morphisms are module homomorphisms (e.g. $f : M \to N$ such that $f(rx+y) = f(rx) + f(y) = rf(x) + f(y)$)
- An (R,S)-module or bimodule is an abelian group M which is a left R-module and a right S-module such that for all $r \in R$ and $s \in S$ and $m \in M$

$$(rm)s = r(ms)$$

Algebraic Aside

CLAIM If M, N are R-modules, and M is an (R,S)-module, then R-mod(M, N) has an S-module structure.
PROOF $\forall f \in$ R-mod$(M, N) \forall s, t \in S \forall m \in M$

$$(s \cdot_S f)(rm) = f((rm)s) = f(r(ms)) = rf(xs)$$

and

$$(st \cdot f)(m) = f(mst) = (t \cdot f)(xs) = s \cdot (t \cdot f)(x)$$

follows by setting $(s \cdot f)m := f(ms)$, i.e. $s \cdot f$ is R-linear and a *ring action* and hence R-mod(M, N) may be regarded as an S-module.

Categorical Notions: Functors

(Covariant) Functors $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ are maps between C and D such that

- for all $X \in |\mathsf{C}|$, $\mathcal{F}(X) \in |\mathsf{D}|$
- for all $f \in \mathsf{C}$, $\mathcal{F}(f) \in \mathsf{D}$
- for all $X \in |\mathsf{C}|$ and composable $f, g \in \mathsf{C}$,

$$\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$$

and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$

Example

The identity functor takes $X \in |\mathsf{C}|$ to X and $f \in \mathsf{C}$ to f

Example

We may consider any set X to be a functor $\mathcal{X} : 1 \to \mathsf{Sets}$.

Functor Properties

- A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is full when for every $X, Y \in |\mathsf{C}|$, the mapping on arrows $\mathcal{X}, \mathcal{Y} : \mathsf{C}(\mathcal{X}, \mathcal{Y}) \to \mathsf{D}(\mathcal{F}(\mathcal{X}), \mathcal{F}(\mathcal{Y}))$ is surjective
- $\mathcal{F}$ is faithful if each $\mathcal{F}_{X,Y}$ is injective
- $\mathsf{C} \subseteq \mathsf{D}$, ie. C is a subcategory of D, if $|\mathsf{C}| \subseteq |\mathsf{D}|$ and $\mathsf{C}(X, Y) \subset \mathsf{D}(X, Y)$ for all $X, Y \in |\mathsf{C}|$, and composition in C is a restriction of composition in D
- A subcategory $\mathsf{C} \subseteq \mathsf{D}$ is broad when $|\mathsf{C}| = |\mathsf{D}|$.

- Given a functor $p : \mathsf{E} \to \mathsf{B}$, we can define a *new* category

with respect to every object in the image of p. Let $I \in |\mathsf{B}|$, and define $\mathsf{E}_I := p^{-1}(I)$ such that

1. objects are $X \in |\mathsf{E}|$ such that $p(X) = I$
2. morphisms are $f \in \mathsf{E}(X, Y)$ such that $p(f) = 1_I \in \mathsf{B}$

- E_I is the fibre category over I
- We say that $X \in |\mathsf{E}_I|$ is above I and similarly $f \in \mathsf{E}$ such that $p(f) = u$ is said to be above u.

Functors: Fibrations

- Given a functor $p : \mathsf{E} \to \mathsf{B}$, we say $f \in \mathsf{E}$ is Cartesian over and $u \in \mathsf{B}(I, J)$ if $p(f) = u$ and for every $g \in \mathsf{E}(Z, Y)$ such that $p(g) = u \circ w$ for some $w \in \mathsf{B}(p(Z), I)$ there is a uniquely determined $h \in \mathsf{E}(Z, X)$ *above* w with $f \circ h = g$
- $f \in \mathsf{E}(X, Y)$ is a Cartesian if it is Cartesian over its underlying map $p(f)$.
- p is a fibration if for every $Y \in |\mathsf{E}|$ and $u \in \mathsf{B}(I, p(Y))$ there is a cartesian morphism $f \in \mathsf{E}(X, Y)$ over u
- Practically understood, fibrations capture *indexing* and *substitution*

Fibrations Example

Let $I \in |\mathrm{Sets}|$.

- The fibre category Pred_I is the subcategory of predicates on I identified with the poset category $(\mathtt{P}(P)(I), \subseteq)$ ordered by inclusion
- Given any $u \in \mathrm{Sets}(I, J)$ we can define a substitution functor $u^* : \mathtt{P}(P)(J) \to \mathtt{P}(P)(I)$ via

$$(Y \subseteq J) \mapsto (\{i \mid u(i) \in Y\} \subseteq I)$$

- If $u = \pi : I \times J \to I$, then π^* is called weakening as it is given by $X \mapsto \{(i, j) \mid i \in X \wedge j \in J\}$ by adding a dummy variable $j \in J$ to the predicate X
- If $u = \Delta : I \to I \times I$, then Δ^* is called contraction as it is given by $\mathtt{P}(P)(I \times I) \ni Y \mapsto \{i \in I \mid (i, i) \in Y\}$, and thus replaces two variables of I with a single variable.

Functors: Examples

- Continuing our work on Pred, we can also define quantifiers as follows: For any $X \subseteq I \times J$
  - $\exists(X) := \{i \in I \mid \exists j \in J, (i, j) \in X\}$ which is a subset of I
  - $\forall(X) := \{i \in I \mid \forall j \in J, (i, j) \in Y)$ which is also a subset of I
  - The assignments given by the quantifiers are functorial on $\mathtt{P}(P)(I \times J) \to \mathtt{P}(P)(I)$
- We can also capture the notion of equality using the diagonal Δ by defining for $\mathtt{Eq}(X) := \Delta(X) = \{(i, j) \mid I \times I \mid i = j \wedge i \in X\}$

Functors: Examples

Continuing our work on Pred and Rel we can also catpure notions of truth, comprehension, and quotients via functors

- (Truth) We can also capture the notion of truth by defining $\top$: Sets $\to$ Pred by assignign each set X the 'truth' predicate $X \subseteq X$ on X.
- (Comprehension) Let $\{-\}$: Pred $\to$ Sets be defined by sending $(X \subseteq I) \mapsto X$
- (Quotients) Let $\mathcal{Q}$: Rel $\to$ Sets be defined by taking $R \mapsto I/\bar{R}$ where $\bar{R}$ is the least equivalence relation $\equiv$ on I such that $R \subseteq \equiv$.
- The quotient construction is formed by pullbacks

Functors: Examples

- A graph and category are small if the collection of objects and arrows are both sets
- Every small graph $\mathcal{G}$ may be viewed as a functor from $\bullet_1 \overset{\rightarrow}{\rightarrow} \bullet_2$ to Sets such that
  - $\mathcal{G}(\bullet_1) := \mathcal{E} \in$ Sets
  - $\mathcal{G}(\bullet_2) : \mathcal{V} \in$ Sets
  - $\mathcal{G}(s) = \sigma : \mathcal{E} \to \mathcal{V}$ and $\mathcal{G}(t) = \tau : \mathcal{E} \to \mathcal{V}$

Functors: Examples

- Let Grp be the category of groups. Let $\mathcal{U} : \text{Grp} \to \text{Sets}$ the functor which sends each group to its underlying set and homomorphism to its underlying set function. This is called the forgetful functor.
- We could also let $\mathcal{U} : \text{Pred} \to \text{Sets}$ be defined naturally by sending each predicate to its underlying set.
- If C is locally small, we can define a contravariant functor $\mathrm{C}(-, X)$ which sends objects Y to the set of arrows $\mathrm{C}(Y, X)$ and arrows $f : Y \to Z$ to $\mathrm{C}(f, 1_X) : \mathrm{C}(Z, X) \to \mathrm{C}(Y, X)$

The Dual Functor

- For any category C, there is a functor $-^{op} : \mathsf{C} \to \mathsf{C}$ which fixes $|\mathsf{C}|$ pointwise, and sends $f \in \mathsf{C}(X, Y)$ to $f^{op} \in \mathsf{C}(Y, X)$
- In particular for any $(g \circ f) : X \to Y \to Z$, $(g \circ f)^{op} = f^{op} \circ g^{op} : Z \to Y \to X$
- Intuitively, this functor reverses the direction of the arrows in each commuting diagram in a category by reversing the source and targets of each map.
- A functor from A^{op} to B is called a contravariant functor

Representative Functors

Functors: Extension By Scalars, pt 1

Given $f \in \text{CRing}(R, S)$, we may define

$$f^! : \text{R-Mod} \to \text{S-Mod}$$

by setting $f^!(M) := \text{R-Mod}(S, M)$, with S-action $s\alpha(s') := \alpha(ss')$ for each $\alpha \in \text{R-Mod}(S, M)$.

Functors: Restriction By Scalars

- Let $f \in \text{CRing}(R, S)$ and $N \in |\text{S-mod}|$. We define

restriction by scalars to be the functor

$$f_* : \text{S-mod} \to \text{R-mod}$$

which fixes $M \in |\text{S-mod}|$ by pre-composing the S-action σ on M with $f : R \to S$, so that M is now an R-module, i.e. $rm := f(r)m$ where $f(r) \in \text{End}M$.

- This is covariantly functorial, and when f is injective, we may naturally view R as a subring of S, whence the name *restriction* by scalars.
- Since kernels and images of module homomorphisms are the same regardless of the base ring, restriction by scalars is trivially exact.
- S is a natural R-module, with the action of r on S given by the multiplication of $f(r)s$ in S.

- Suppose that $M, N \in |\text{R-Mod}|$, then the tensor product $M \otimes_R N$ is an R-module given by an R-bilinear map $\otimes : M \times N \to M \otimes_R N$ such that every other R-bilinear $\varphi : M \times N \to P$ will factor *uniquely* through $M \otimes_R N$, e.g.

$$\exists! \bar{\varphi}, (\varphi = \bar{\varphi} \circ \otimes)$$

- For all $N \in |\text{R-Mod}|$, $R \otimes_R N \cong N$, as every R-bilinear $R \times N \to P$ factors through N by setting $\otimes(r, n) = rn$.
- By the *uniqueness property* of universal objects, we have our natural isomorphism $N \cong R \otimes_R N$
- We can define a covariant functor \$\

Lemma: If $M \in |\text{R-Mod}|$ and N is (R,S)-bimodule, and $P \in |\text{S-Mod}|$, then $\text{R-Mod}(M, \text{S-Mod}(N, P)) \cong \text{S-Mod}(M \otimes_R N, P)$

- For every $\alpha \in \text{R-Mod}(M, \text{S-Mod}(N, P))$, there is a $\varphi : M \times N \to P$ determined via $\varphi(m, -) : \alpha(m)$
- Such φ are \$Z\$-bilinear, and for any $r \in R, m \in M, n \in N$

$$\varphi(rm, n) = \alpha(rm,)(n) =_1 r\alpha(m)(n) = \alpha(m)(rn) = \varphi(m, rn)$$

Functors: Extension By Scalars, pt 2

Given $\mathcal{F}$:D→C\$ and an object $C \in |\mathrm{C}|$, we denote by $A/\mathcal{F}$ the comma category whose

- objects are the arrows $f : C \to \mathcal{F}(Y)$, denoted by (f, Y)
- morphisms are the arrows $h \in \mathrm{D}(Y, Z)$ such that $\mathcal{F}(h) \circ f = f'$, i.e. $\langle \in A/\mathcal{F}((f, Y), (f, Z))$
- If $\mathcal{F} := 1_{\mathrm{C}}$, then $C/\infty_{\mathrm{C}} := C/\mathrm{C}$.
- Denote by \$C/C:=(C/C$^{\mathrm{op}}$)$^{\mathrm{op}}$. This is the co-slice category.

Isomorphism

- Inside categories, objects are unique up to *isomorphism.
- An arrow $f \in \mathsf{C}(X, Y)$ is an isomorphism if there is a $g \in \mathsf{C}(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$; we say X is isomorphic to Y if such f, g exist.
- If $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} : \mathsf{D} \to \mathsf{C}$ such that $\mathcal{F} \circ \mathcal{G} = 1_{\mathsf{D}}$ and $\mathcal{G} \circ \mathcal{F} = 1_{\mathsf{C}}$, then C is isomorphic to D

Presheaves

+Given two functors $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$, a natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is a *family of morphisms* \$$(\eta_x)_{|\mathsf{C}|}$ indexed by $|\mathsf{C}|$, such that for each $f \in \mathsf{C}(X, Y)$,

$$\mathcal{G}(f) \circ \eta_X = \eta_Y \circ \mathcal{F}(f)$$

- Given any two categories C, D, we can define a functor category D^{C} whose objects are functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations

Revisiting Grph

Let Grph be the category of graphs

- objects are functors $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \tau)$ such that $\mathcal{V}(\mathcal{G})$ are the vertices of $\mathcal{G}$ and $\mathcal{E}(\mathcal{G})$ are the directed edges;
- morphisms graph homomorphisms $F : \mathcal{G} \Rightarrow \mathcal{H}$ are pairs of mappings
- $F : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$
- $F : \mathcal{E}(\mathcal{G}) \to \mathcal{E}(\mathcal{G})$ such that for $f : X \to Y$, $F(f) : F(X) \to F(Y)$.

Yoneda Lemma: If C is locally small, and $\mathcal{F} : C^{op} \to$ Sets, then $\text{Sets}^{C^{op}}(C(-, X), \mathcal{F}) \cong \mathcal{F}(X)$

Denote $C(-, X) := h_X$. For any $\eta : h_X \Rightarrow \mathcal{F}$, we obtain $\eta_X(1_X) \in \mathcal{F}(X)$.
On the other hand, given $x \in \mathcal{F}(X)$, we obtain natural transformation $\check{x} : h_X \to \mathcal{X}$ by setting $\check{x} : C(Y, X) \Rightarrow \mathcal{F}(X)$ such that $g : Y \to X$ is sent to $\mathcal{F}(g)(x)$.
These maps are inverse to one another.